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Contractions of Lie algebras and the separation of variables: interbase expansions

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Abstract

Lie algebra contractions from $o(n+1)$ to $e(n)$ are used to obtain asymptotic limits of interbase expansions between bases corresponding to different subgroup chains for the group $O(n+1)$. The contractions lead to interbase expansions for different subgroup chains of the Euclidean group $E(n)$. They provide asymptotic formulae for quantities such as Wigner rotation matrices, Clebsch–Gordan coefficients and Racah coefficients.

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1. Introduction

This paper is the third in a series [1, 2] devoted to contractions of rotation groups $O(n+1)$ to Euclidean groups $E(n)$ and the separation of variables in Laplace–Beltrami equations. In the first paper [1] we considered the sphere S_2 on which the equation

$$\Delta_{LB}\Psi = -\lambda\Psi \qquad \Delta_{LB} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i}\sqrt{g}\,g^{ik}\frac{\partial}{\partial\xi_k} \qquad g = \det g_{ik} \tag{1.1}$$

allows the separation of variables in two coordinate systems: spherical and elliptic ones. The contraction parameter was the radius R of the sphere. For $R \to \infty$ the sphere $S_n \sim O(n+1)/O(n)$ goes into the Euclidean space $E_n \sim E(n)/O(n)$. For $n = 2$ the two separable coordinate systems on S_2 go into four separable coordinate systems on E_2, namely Cartesian, polar, parabolic and elliptic ones. Depending on how the limit is taken, spherical coordinates go into polar or Cartesian ones. Elliptic coordinates on S_2 go into elliptic or parabolic coordinates on E_2. Via a two-step procedure, through spherical coordinates, they also contract to Cartesian and polar coordinates on E_2. The contraction was followed through on several levels: the coordinates, the complete sets of commuting operators, the separated equations and the eigenfunctions and eigenvalues.

In the second paper [2] the dimension of the space was arbitrary, but only the simplest types of coordinates were considered, namely subgroup-type coordinates. These are associated with chains of subgroups of $O(n+1)$, or $E(n)$, respectively.

Vilenkin, Kuznetsov and Smorodinsky [3, 4] developed a graphical method, the 'method of trees' to describe subgroup-type coordinates on S_n. The corresponding separated eigenfunctions are hyperspherical functions (also called polyspherical functions) [5–8]. Their relation to subgroup chains and subgroup diagrams was analysed in [2], as were their contractions to subgroup-type separated basis functions for the groups $E(n)$.

In many-body theories it is often necessary to expand one type of hyperspherical functions in terms of other ones. The expansion coefficients have been called T-coefficients, or overlap functions. The corresponding coefficients for functions on S_n were calculated by Kil'dyushov [7].

The purpose of this paper is to study the $R \to \infty$ contraction limit of the interbase expansions and overlap functions for the different spherical and hyperspherical functions on S_n. The mathematical motivation is to obtain asymptotic limits of various expansions and of the overlap functions. These are objects of considerable physical interest: Wigner rotation matrices, Clebsch–Gordan coefficients, Racah coefficients, etc. The physical motivation goes back to the original work of Inönü and Wigner [9]. Typically, a Lie group, or Lie algebra contraction relates two different theories. The contraction parameter in our case is not the speed of light, so we are not relating relativistic and non-relativistic theories. Rather, we are relating theories in flat and curved spaces, or theories of spherical and highly elongated objects, e.g. nuclei [10].

The contractions we use are analytical ones: the radius of the sphere is built into the infinitesimal operators and into the sets of commuting operators, not only into the structure constants. The contractions can be viewed as singular changes of bases, as was the case of the original Inönü–Wigner ones. They are also 'graded contractions' [11, 12], in this case corresponding to a Z_2 grading of $o(3)$, $o(4)$ and more generally $o(n+1)$.

The overall point of view of the separation of variables that we are taking is an operator one [13–18]. Thus, let G be the isometry group of the considered Riemannian or pseudo-Riemannian space and L be its Lie algebra. Let $\{X_1, X_2, \ldots, X_N\}$ be a basis of L and

$$Y_a = \sum_{ik} A^a_{ik} X_i X_k \qquad [Y_a, Y_b] = 0 \qquad A^a_{ik} = A^a_{ki} \tag{1.2}$$

a complete set of commuting second-order operators in the enveloping algebra of L. The separated eigenfunctions will be the common eigenfunctions of such a complete set

$$Y_a \Psi = -\lambda_a \Psi \qquad \Psi = \prod_i^n f_i(\xi_i) \tag{1.3}$$

where ξ_i are the separable coordinates. For subgroup-type coordinates all the operators Y_a are Casimir operators of subalgebras of L (the Laplace–Beltrami operator Δ_{LB} is included in the set $\{Y_a\}$).

The paper is organized as follows. In section 2 we give a brief review of the method of trees and present a general formula for the overlap functions. The results are known, but we include them to make the paper readable independently. In section 3 we introduce the analytical contractions from S_n to E_n and give contraction formulae for basis functions. The essentially new material starts in section 4 where we relate overlap functions for the sphere S_n with those for the Euclidean space E_n. In sections 5 and 6 we specialize the general results to low-dimensional cases, namely S_2 and S_3, when the formulae become simple and explicit. Some conclusions are drawn in the final section 7. Special cases of the T-function are given in the appendix.

2. Method of trees and overlap functions

The method of trees [3–7] for the rotation group $O(n+1)$ and the sphere $S_n \sim O(n+1)/O(n)$ was briefly reviewed in our earlier paper [2]. To make this paper readable independently, let us repeat some basic facts. We are dealing with the Laplace–Beltrami equation on an n-dimensional sphere S_n (1.1) and use a graphical method, the 'methods of trees', for characterizing different types of subgroup coordinates, or hyperspherical coordinates on S_n. These methods are best presented in the original article [3] and in the books [4–6].

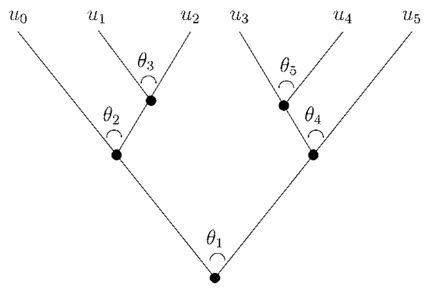

Figure 1. Example of tree for polyspherical coordinates on the sphere S_5.

Let us briefly describe the method of trees (see figure 1). Each end point $u_i, i = 0, 1, 2, \ldots, n$ on the tree corresponds to a Cartesian coordinate in the ambient space E_{n+1}. At each branching point, we introduce an angle θ_j. We move along the tree from the ground upwards to a specific coordinate u_i. At each branching point, we write $\cos\theta_j$, if we go to the left, and $\sin\theta_j$, if we go to the right. The coordinate u_i is represented as a product of the trigonometric functions obtained when following branches leading from the bottom of the tree to the end point u_i. For example, to the tree in figure 1 there correspond the following polyspherical coordinates:

$$u_0 = R\cos\theta_1\cos\theta_2 \qquad u_1 = R\cos\theta_1\sin\theta_2\cos\theta_3$$
$$u_2 = R\cos\theta_1\sin\theta_2\sin\theta_3 \qquad u_3 = R\sin\theta_1\cos\theta_4\cos\theta_5$$
$$u_4 = R\sin\theta_1\cos\theta_4\sin\theta_5 \qquad u_5 = R\sin\theta_1\sin\theta_4.$$

To each branching point on the tree diagram we also associate quantum numbers l_j. These will determine the eigenvalue λ_j of the $O(k)$ Laplace–Beltrami operators according to the formula

$$Y_j\Psi = R^2\Delta_{LB}^k\Psi = \lambda_j\Psi \qquad \lambda_j = l_j(l_j + k - 2) \tag{2.1}$$

where k is the dimension of the ambient space above the corresponding vertex on the tree. For $k = 2$ we have $l_j = 0, \pm1, \pm2, \ldots,$ for $k \geqslant 3$ the eigenvalues λ_j are non-negative integers. To specify the separated wavefunction

$$\Psi = \Pi_{j=1}^n \Psi_j(\theta_j) \tag{2.2}$$

on S_n, we follow [3–6] and introduce four types of vertices, or 'cells' on a tree, as illustrated in figures 2(*a*)–(*c*). A full circle denotes a 'closed' end, otherwise the end is 'open'. An open end leads directly to a coordinate in the ambient space. A closed one leads to further branches. Each vertex and each angle θ_j provides a 'building block' $\Psi_j(\theta_j)$ for the wavefunction $\Psi(\theta_1, \ldots, \theta_n)$. Specifically, we have

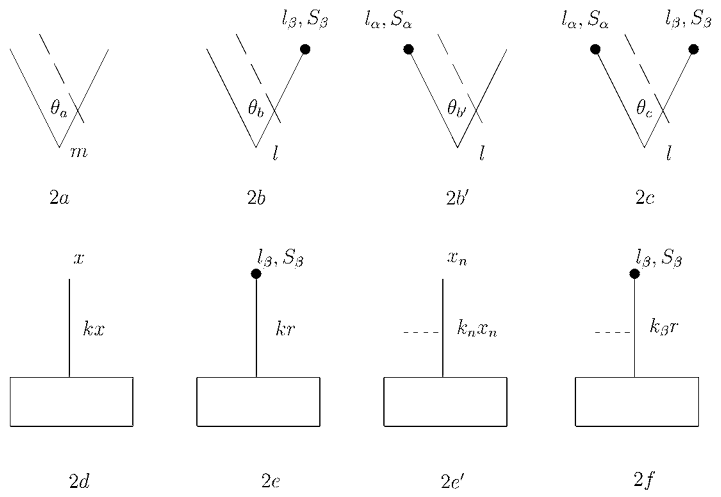

Figure 2. Elementary cells for S_n (diagrams $2a, \ldots, 2c$) and their contractions to E_n ones (diagrams $2d, \ldots, 2f$). Full circles correspond to closed ends. There are S_α further vertices above the vertex alpha. The broken lines are explained in the text.

Cell of type 2a.

$$\Psi_m(\theta_a) = \frac{1}{\sqrt{2\pi}} e^{im\theta_a} \qquad m = 0, \pm 1, \pm 2, \ldots \qquad 0 \leqslant \theta_a < 2\pi. \tag{2.3}$$

Cell of type 2b.

$$\begin{aligned} &\Psi^c_{n,l_\beta}(\theta_b) = N_n^{c,c} (\sin\theta_b)^{l_\beta} P_n^{(c,c)}(\cos\theta_b) \\ &n = l - l_\beta \qquad c = l_\beta + \tfrac{1}{2} S_\beta \qquad n = 0, 1, 2, \ldots \qquad 0 \leqslant \theta_b \leqslant \pi \end{aligned} \tag{2.4}$$

where $P_n^{(a,b)}(x)$ are the Jacobi polynomials.

Cell of type 2b′.

$$\begin{aligned} &\Psi^a_{n,l_\alpha}(\theta_{b'}) = N_n^{a,a} (\cos\theta_{b'})^{l_\alpha} P_n^{(a,a)}(\sin\theta_{b'}) \\ &n = l - l_\alpha \qquad a = l_\alpha + \tfrac{1}{2} S_\alpha \qquad n = 0, 1, 2, \ldots \qquad -\pi/2 \leqslant \theta_{b'} \leqslant \pi/2. \end{aligned} \tag{2.5}$$

Cell of type 2c.

$$\begin{aligned} &\Psi^{b,a}_{n,l_\beta,l_\alpha}(\theta_c) = 2^{(b+a)/2+1} N_n^{b,a} (\sin\theta_c)^{l_\beta} (\cos\theta_c)^{l_\alpha} P_n^{(b,a)}(\cos 2\theta_c) \\ &n = \tfrac{1}{2}(l - l_\alpha - l_\beta) \qquad b = l_\beta + \tfrac{1}{2} S_\beta \\ &a = l_\alpha + \tfrac{1}{2} S_\alpha \qquad n = 0, 1, 2, \ldots \qquad 0 \leqslant \theta_c \leqslant \pi/2. \end{aligned} \tag{2.6}$$

Here, S_α and S_β are the numbers of vertices above the vertex l_α and l_β, respectively. The normalization constants are

$$N_n^{a,b} = \left\{ \frac{(2n + a + b + 1)\Gamma(n + a + b + 1)n!}{2^{a+b+1}\Gamma(n + a + 1)\Gamma(n + b + 1)} \right\}^{1/2}.$$

We mention that the wavefunctions (2.4) and (2.5) can also be expressed in terms of the Gegenbauer polynomials by using the formula [19]

$$C_n^\lambda(x) = \frac{\Gamma(2\lambda + n)\Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda)\Gamma(\lambda + n + \frac{1}{2})} P_n^{(\lambda-1/2,\lambda-1/2)}(x).$$

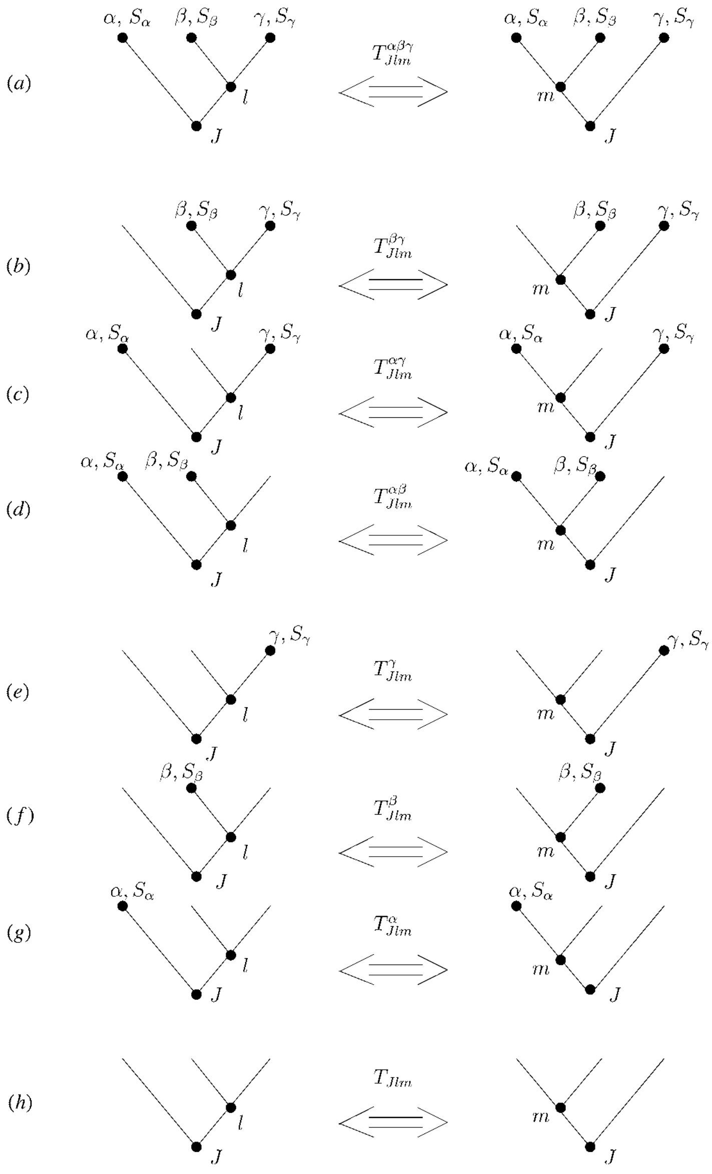

Figure 3. Diagrams representing elementary 'transitions' between trees.

A convenient way of calculating the T-coefficients corresponding to a transformation from one $SO(n+1)$ tree to another, is to introduce a sequence of '*elementary*' trees, each differing from the previous one by the transplantation of exactly one branch from one side of a branching point to the other. The general T-matrix will be factorized into a product of '*elementary T-matrices*' corresponding to such elementary transformations. Each elementary T-matrix connects two tree-type diagrams. Both are cells with three ends, each of which can be either open or closed (see figure 3). Eight inequivalent elementary diagrams of this type exist: one with three closed ends, three with two closed ends, three with one closed end and one with all three ends open (see figure 3). The T-coefficients for all eight types of elementary transformations were calculated by Kil'dyushov [7]. They were expressed in terms of generalized hypergeometric functions of argument $x = 1$: ${}_3F_2(1)$, ${}_4F_3(1)$, Wigner D-functions, or Clebsch–Gordon and Racah coefficients for positive discrete series of representations of the group $SU(1,1)$ [20]. We mention that a relation between the T-coefficients and polynomials of discrete variables has been established [21], namely, the Racah–Wilson, Hahn and Krawtchouk polynomials.

The T-coefficient, representing the *general transformation*, corresponds to the diagram with three closed ends in figure 3(a) are [7]

$$
\begin{aligned}
T^{\alpha\beta\gamma}_{Jlm} = {}& \frac{\sqrt{(l+\frac{1}{2}(S_\beta+S_\gamma)+1)(m+\frac{1}{2}(S_\alpha+S_\beta)+1)(\frac{1}{2}(J-m-\gamma))!}}{\Gamma(\beta+\frac{1}{2}S_\beta+1)} \\
& \times\frac{\Gamma(\frac{1}{2}(J-\alpha-\beta+\gamma+S_\gamma)+1)}{\Gamma(\frac{1}{2}(J+\alpha+\beta-\gamma+S_\alpha+S_\beta)+2)}\Big\{\big[\Gamma(\tfrac{1}{2}(l+\beta+\gamma+S_\beta+S_\gamma)+1) \\
& \times\Gamma(\tfrac{1}{2}(l+\beta-\gamma+S_\beta)+1)\Gamma(\tfrac{1}{2}(J+\alpha+l+S_\alpha+S_\beta+S_\gamma)+2)\big] \\
& \times\big[\Gamma(\tfrac{1}{2}(m+\alpha-\beta+S_\alpha)+1)\Gamma(\tfrac{1}{2}(l-\beta+\gamma+S_\gamma)+1) \\
& \times\Gamma(\tfrac{1}{2}(J+m+\gamma+S_\alpha+S_\beta+S_\gamma)+2)\big]^{-1}\Big\}^{1/2} \\
& \times\Big\{\big[\Gamma(\tfrac{1}{2}(J+\alpha-l+S_\alpha)+1)\Gamma(\tfrac{1}{2}(J+m-\gamma+S_\alpha+S_\beta)+2) \\
& \times\Gamma(\tfrac{1}{2}(m+\alpha+\beta+S_\alpha+S_\beta)+1)\Gamma(\tfrac{1}{2}(m-\alpha+\beta+S_\beta)+1)\big] \\
& \times\big[(\tfrac{1}{2}(m-\alpha-\beta))!(\tfrac{1}{2}(l-\beta-\gamma))!(\tfrac{1}{2}(J-l-\alpha))! \\
& \times\Gamma(\tfrac{1}{2}(J+l-\alpha+S_\beta+S_\gamma)+2)\Gamma(\tfrac{1}{2}(J-m+\gamma+S_\gamma)+1)\big]^{-1}\Big\}^{1/2} \\
& \times{}_4F_3\left\{\left.\begin{matrix} -\frac{1}{2}(m-\alpha-\beta), \frac{1}{2}(m+\alpha+\beta+S_\alpha+S_\beta)+1, \\ \frac{1}{2}(l-\gamma+\beta+S_\beta)+1, -\frac{1}{2}(l-\beta+\gamma+S_\gamma); \\ \beta+\frac{1}{2}S_\beta+1, \frac{1}{2}(J-\gamma+\alpha+\beta+S_\alpha+S_\beta)+2, \\ -\frac{1}{2}(J-\alpha-\beta+\gamma+S_\gamma) \end{matrix}\right| 1\right\}.
\end{aligned}
\tag{2.7}
$$

Here, for brevity we replace the numbers $(l_\alpha, l_\beta, l_\gamma)$ by (α, β, γ) and S_{α_j} $(\alpha_j = \alpha, \beta, \gamma)$ is the number of branching points above the point α_j.

It was pointed out in [21, 22] that if we allow the numbers $S_{\alpha_j} = -1$, $\alpha_j = 0, 1$ we obtain the T functions for diagrams with open ends. The transition matrices for all eight types of T-'cells' in figure 3 (up to a phase factor!), can be obtained in this manner and they are presented in the appendix.

3. Contractions of the basis functions

Let us introduce a standard basis $L_{\mu\nu}$ for the Lie algebra $o(n+1)$

$$L_{ik} = (u_i\partial_k - u_k\partial_i) \tag{3.1}$$

$$[L_{ik}, L_{mn}] = \delta_{km}L_{in} + \delta_{in}L_{km} - \delta_{im}L_{kn} - \delta_{kn}L_{im} \qquad 0 \leqslant i, k, m, n \leqslant n. \tag{3.2}$$

The Laplace–Beltrami operator on S_n is

$$\Delta_{LB} = \frac{1}{R^2}\sum_{0\leqslant i<k\leqslant n} L_{ik}^2. \tag{3.3}$$

We shall use R^{-1} as a contraction parameter. To realize the contraction explicitly, let us introduce Beltrami coordinates on the sphere S_n putting

$$y_i = R\frac{u_i}{u_0} = u_i\left(1 - \frac{1}{R^2}\sum_{k=1}^{n} u_k^2\right)^{-1/2} \qquad i = 1, 2, 3, \ldots, n. \tag{3.4}$$

Then, the $O(n+1)$ generators can be expressed as

$$\frac{L_{0i}}{R} \equiv \pi_i = p_i + \frac{y_i}{R^2}\sum_{k=1}^{n}(y_k p_k) \tag{3.5}$$

$$L_{ik} \equiv y_i p_k - y_k p_i = y_i \pi_k - y_k \pi_i \qquad i, k = 1, 2, \ldots, n \tag{3.6}$$

where $p_i = \partial/\partial y_i$. The commutation relations are now

$$[L_{ik}, L_{mn}] = \delta_{km}L_{in} + \delta_{in}L_{km} - \delta_{im}L_{kn} - \delta_{kn}L_{im} \tag{3.7}$$

$$[\pi_i, L_{kj}] = \delta_{ik}\pi_j - \delta_{ij}\pi_k \qquad [\pi_i, \pi_k] = \frac{L_{ik}}{R^2} \tag{3.8}$$

so that as $R \to \infty$ the $o(n+1)$ algebra contracts to the Euclidean $e(n)$ one. The Beltrami coordinates y_i (3.4) contract to the Cartesian coordinates on E_n, and we have

$$y_i \to x_i \qquad \pi_i \to p_i = \frac{\partial}{\partial x_i} \tag{3.9}$$

so that the rotation generators L_{0i} turn into the translations p_i. The $o(n+1)$ Laplace–Beltrami operator (3.3) contracts to the $e(n)$ one

$$\Delta_{\mathrm{LB}} = \sum_{i=1}^{n}\pi_i^2 + \sum_{i,k=1}^{n}\frac{L_{ik}^2}{2R^2} \to \Delta = p_1^2 + p_2^2 + \cdots + p_n^2. \tag{3.10}$$

Recently [2], we introduced a *graphical method* of connecting subgroup-type coordinates systems on the sphere S_n (characterized by tree diagrams) and on the Euclidean space E_n (characterized by cluster diagrams) and gave the rules relating the contraction limit $R \to \infty$ of the coordinates, invariant operators, eigenvalues and basis functions.

Graphically, the contraction $R \to \infty$ can be interpreted as cutting off the branch from the ground to the point u_0 along the broken line, represented for the general S_n tree diagram in figure 4(a). The broken line then becomes the ground for the corresponding E_n cluster diagram of figure 4(b) (see also figure 2). The ambient space coordinates $(u_0, u_1, \ldots, u_n)$ for S_n are transformed into the Cartesian coordinates $(x_1, x_2, \ldots, x_n)$. The angles $\theta_1, \theta_2, \ldots, \theta_j$ and the angular momentum quantum numbers $l_1, l_2, \ldots, l_j$ leading to the branches, cut off by the dotted line, satisfy $\theta_i \to 0$ and $l_i \to \infty$. In the contraction the angles are replaced by the radial coordinates r_i, or Cartesian coordinates x_m (if a surviving branch leads directly to a single coordinate on S_n and E_n). The angular momenta l_i are replaced by some constants k_i. We have

$$\theta_j \sim \frac{r_j}{R} \qquad l_j \sim k_j R \quad R \to \infty. \tag{3.11}$$

When we cut off the branches of a tree as in figure 4(a), the cutting line intersects an elementary cell at a branch (see figure 2) and each elementary cell in the top row of figure 2 goes into an elementary trunk, as indicated in the bottom row of figure 2. The limiting procedure for cells is always the same as in equation (3.11).

Let us now run through the contraction of basis functions for the individual cells in figure 2.

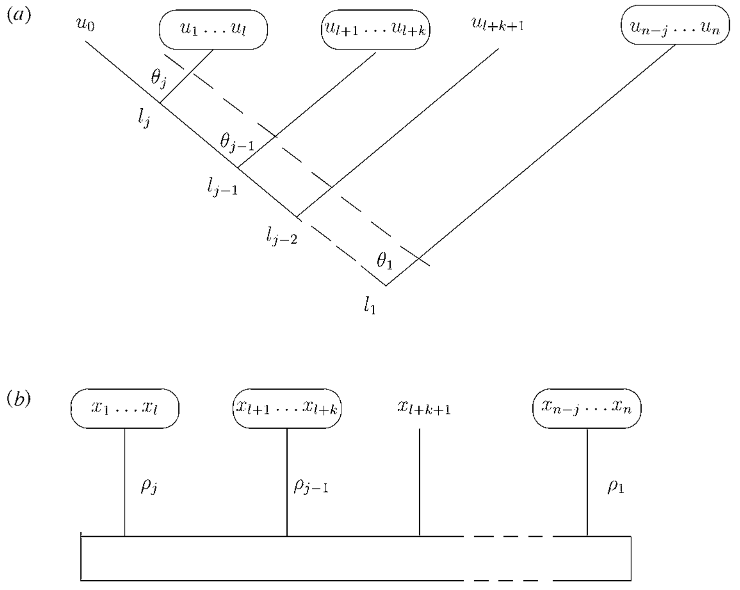

Figure 4. Contractions of tree diagrams into cluster ones: from S_n to E_n.

Cells 2a–d. In the contraction limit $R \to \infty, m \sim kR, \theta_a \sim x/R$ we have

$$\Phi_k(x) = \lim_{R\to\infty} \Psi_m(\theta_a) = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}m\theta_a} = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}kx}. \tag{3.12}$$

Cells 2b–e. In the contraction limit $l \sim kR, \theta_b \sim r/R$ we have

$$\Phi^c_{k,l_\beta}(r) = \lim_{R\to\infty} \frac{1}{\sqrt{R^{S_\beta+1}}} \Psi^c_{n,l_\beta}(\theta_b) = \lim_{R\to\infty} \frac{N^{l_\beta+\frac{1}{2}S_\beta, l_\beta+\frac{1}{2}S_\beta}_{l-l_\beta}}{\sqrt{R^{S_\beta+1}}} (\sin\theta_b)^{l_\beta} P^{(l_\beta+\frac{1}{2}S_\beta, l_\beta+\frac{1}{2}S_\beta)}_{l-l_\beta}(\cos\theta_b)$$

$$= \sqrt{\frac{k}{r^{S_\beta}}} J_{l_\beta+\frac{1}{2}S_\beta}(kr) \tag{3.13}$$

where $J_\nu(z)$ is a Bessel function.

Cells 2b′–e′. In the limit $R \to \infty$ and $\theta_{b'} \sim x_n/R, l \sim kR, l_\alpha \sim pR$ we have

$$\Phi^a_{k,p}(x_n) = \lim_{R\to\infty} (-1)^{\frac{1}{2}(l-l_\alpha)} \Psi^a_{n,l_\alpha}(\theta_{b'}) = \lim_{R\to\infty} (-1)^{\frac{1}{2}(l-l_\alpha)} N^{l_\alpha+\frac{1}{2}S_\alpha, l_\alpha+\frac{1}{2}S_\alpha}_{l-l_\alpha} (\cos\theta_{b'})^{l_\alpha}$$

$$\times P^{(l_\alpha+\frac{1}{2}S_\alpha, l_\alpha+\frac{1}{2}S_\alpha)}_{l-l_\alpha}(\sin\theta_{b'}) = \sqrt{\frac{2k}{\pi k_n}} \begin{cases} \cos(k_n x_n) & (l-l_\alpha) \text{ even} \\ -\mathrm{i}\sin(k_n x_n) & (l-l_\alpha) \text{ odd} \end{cases} \tag{3.14}$$

where $k^2 = k_n^2 + p^2$.

Cells 2c–f. In the limit $R \to \infty$ $l \sim kR, l_\alpha \sim k_\alpha R$ and $\theta_c \sim r/R$, we have

$$\Phi^{l_\beta}_{k,k_\beta,k_\alpha}(r) = \lim_{R\to\infty} \frac{\Psi^{b,a}_{n,l_\beta,l_\alpha}(\theta_c)}{\sqrt{R^{S_\beta+1}}} = \lim_{R\to\infty} \sqrt{\frac{2^{l_\alpha+l_\beta+(S_\alpha+S_\beta)/2+2}}{R^{S_\beta+1}}} N^{l_\beta+\frac{1}{2}S_\beta, l_\alpha+\frac{1}{2}S_\alpha}_{\frac{1}{2}(l-l_\alpha-l_\beta)} (\sin\theta_c)^{l_\beta} (\cos\theta_c)^{l_\alpha}$$

$$\times P^{(l_\beta+\frac{1}{2}S_\beta, l_\alpha+\frac{1}{2}S_\alpha)}_{\frac{1}{2}(l-l_\alpha-l_\beta)}(\cos 2\theta_c) = \sqrt{\frac{2k}{r^{S_\beta}}} J_{l_\beta+\frac{1}{2}S_\beta}(k_\beta r) \tag{3.15}$$

where $k^2 = k_\alpha^2 + k_\beta^2$ and the parameters a, b and c are determined using formulae (2.4)–(2.6).

Thus, using these contractions for basis functions corresponding to the elementary cells, we go from (2a, ..., 2c) to (2d, ..., 2f) (see figure 2) and can determine the general contractions for hyperspherical functions corresponding to any $O(n+1)$ tree (see figure 4).

4. Contractions of the interbase expansions

Having the explicit form of the T-coefficients for all eight types of 'elementary' transitions between trees, we shall now consider the contraction limit $R \to \infty$ for the interbase expansions in figure 3.

4.1. Contraction of Racah coefficients: three closed ends

The tree on the left-hand side of figure 5(a) corresponds to the subgroup chains $O(n+1) \supset O(n_\alpha + n_\beta) \otimes O(n_\gamma)$, while the tree on the right-hand side of figure 5(a) corresponds to the chain $O(n+1) \supset O(n_\alpha) \otimes O(n_\gamma + n_\beta)$, where $n+1 = n_\alpha + n_\gamma + n_\beta$.

The interbase expansion corresponding to the transformations between trees in figure 5(a) has the form

$$\Psi_{Jm}^{\alpha\beta\gamma}(\theta_1', \theta_2') = \sum_{l=\beta+\gamma,\beta+\gamma+2,\ldots}^{J-\alpha} T_{Jlm}^{\alpha\beta\gamma} \Psi_{Jl}^{\alpha\beta\gamma}(\theta_1, \theta_2) \tag{4.1}$$

where

$$\cos\theta_1 = \cos\theta_1' \cos\theta_2' \qquad \cot\theta_2 = \cot\theta_1' \sin\theta_2'.$$

The T-coefficients are given by formula (2.7), and the wavefunctions Ψ can be obtained with the help of the rules of section 2 (equation (2.6)).

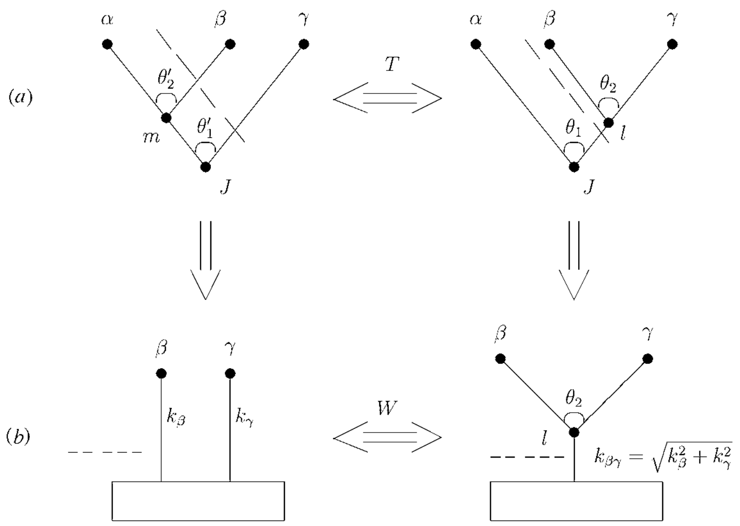

Figure 5. Contractions for three closed ends. A broken line in the bottom row implies that the closed end α gives rise to further trees in the E_n cluster diagram.

Consider now the contraction limit $R \to \infty$ in the interbasis expansion (4.1). For large R we put

$$J \sim kR \qquad m \sim pR \qquad \alpha \sim qR$$
$$\theta_1' \sim \frac{r_\gamma}{R} \qquad \theta_2' \sim \frac{r_\beta}{R} \qquad \theta_1 \sim \frac{r_{\beta\gamma}}{R} \tag{4.2}$$

where $r_{\beta\gamma} = \sqrt{r_\beta^2 + r_\gamma^2}$, $k_\beta^2 = p^2 - q^2$, $k_\gamma^2 = k^2 - p^2$ and $k_{\beta\gamma}^2 = k_\beta^2 + k_\gamma^2$ and we have

$$\lim_{R\to\infty} \Psi_{Jm}^{\alpha\beta\gamma}(\theta_1', \theta_2') = \Phi_{kk_\beta k_\gamma}^{\beta\gamma}(r_\beta, r_\gamma) = \frac{2\sqrt{kp}}{(r_\beta)^{S_\beta/2}(r_\gamma)^{S_\gamma/2}} J_{\beta+\frac{1}{2}S_\beta}(k_\beta r_\beta) J_{\gamma+\frac{1}{2}S_\gamma}(k_\gamma r_\gamma) \tag{4.3}$$

$$\lim_{R\to\infty} \Psi_{Jl}^{\alpha\beta\gamma}(\theta_1, \theta_2) = \Phi_{kk_{\beta\gamma}}^{l\beta\gamma}(r_{\beta\gamma}, \theta_2) = \frac{\sqrt{2k(2l + S_\beta + S_\gamma + 2)}}{(r_{\beta\gamma})^{1+\frac{1}{2}(S_\beta+S_\gamma)}} J_{l+\frac{1}{2}(S_\beta+S_\gamma)+1}(k_{\beta\gamma} r_{\beta\gamma})$$
$$\times \sqrt{\frac{\Gamma(\frac{1}{2}(l + S_\beta + S_\gamma + \beta + \gamma) + 1)(\frac{1}{2}(l - \beta - \gamma) + 1)!}{\Gamma(\frac{1}{2}(l - \beta + \gamma + S_\gamma) + 1)\Gamma(\frac{1}{2}(l + \beta - \gamma + S_\beta) + 1)}}$$
$$\times (\sin\theta_2)^\gamma (\cos\theta_2)^\beta P_{\frac{1}{2}(l-\beta-\gamma)}^{(\gamma+\frac{1}{2}S_\gamma, \beta+\frac{1}{2}S_\beta)}(\cos 2\theta_2). \tag{4.4}$$

Using the asymptotic formulae for the ${}_4F_3$ functions and Γ functions [19] in equation (2.7), we obtain

$$\lim_{R\to\infty} T_{Jlm}^{\alpha\beta\gamma} = W_{kk_{\beta\gamma}k_\beta k_\gamma}^{l\beta\gamma} = \Big\{\big[2p(2l + S_\beta + S_\gamma + 2)\Gamma(\tfrac{1}{2}(l + \beta + \gamma + S_\beta + S_\gamma) + 1)$$
$$\times \Gamma(\tfrac{1}{2}(l + \beta - \gamma + S_\beta + S_\gamma) + 1)\big]$$
$$\times \big[(\tfrac{1}{2}(l - \beta - \gamma))!\Gamma(\tfrac{1}{2}(l - \beta + \gamma + S_\gamma) + 1)[\Gamma(\beta + \tfrac{1}{2}S_\beta + 1)]^2\big]^{-1}\Big\}^{1/2}$$
$$\times \frac{k_\beta^{\beta+\frac{1}{2}S_\beta} k_\gamma^{\gamma+\frac{1}{2}S_\gamma}}{k_{\beta\gamma}^{\beta+\gamma+\frac{1}{2}(S_\beta+S_\gamma)+1}}$$
$$\times {}_2F_1\left(-\tfrac{1}{2}(l - \beta - \gamma), \tfrac{1}{2}(l + \beta + \gamma + S_\gamma + S_\beta) + 1; \beta + \tfrac{1}{2}S_\beta + 1; \frac{k_\beta^2}{k_\beta^2 + k_\gamma^2}\right)$$
$$= (-1)^{\frac{1}{2}(l-\beta-\gamma)}$$
$$\times \sqrt{\frac{2p(2l + S_\gamma + S_\beta + 2)(\frac{1}{2}(l - \beta - \gamma))!\Gamma(\frac{1}{2}(l + \beta + \gamma + S_\beta + S_\gamma) + 1)}{k_{\beta\gamma}^2 \Gamma(\frac{1}{2}(l + \beta - \gamma + S_\beta) + 1)\Gamma(\frac{1}{2}(l - \beta + \gamma + S_\gamma) + 1)}} \tag{4.5}$$
$$\times (\cos\phi)^{\beta+\frac{1}{2}S_\beta} (\sin\phi)^{\gamma+\frac{1}{2}S_\gamma} P_{\frac{1}{2}(l-\beta-\gamma)}^{(\gamma+\frac{1}{2}S_\gamma, \beta+\frac{1}{2}S_\beta)}(\cos 2\phi). \tag{4.6}$$

Taking the contraction limit $R \to \infty$ in (4.1), we obtain ($\theta_2 \equiv \theta$)

$$\Phi_{kk_\beta k_\gamma}^{\beta\gamma}(r_\beta, r_\gamma) = \sum_{l=\beta+\gamma,\beta+\gamma+2,\ldots}^{\infty} W_{kk_{\beta\gamma}k_\beta k_\gamma}^{l\beta\gamma} \Phi_{kk_{\beta\gamma}}^{l\beta\gamma}(r_{\beta\gamma}, \theta). \tag{4.7}$$

Using the orthogonality condition for the Jacobi polynomials

$$\int_0^{k_{\beta\gamma}} W_{kk_{\beta\gamma}k_\beta k_\gamma}^{l\beta\gamma} W_{kk_{\beta\gamma}k_\beta k_\gamma}^{l'\beta\gamma *} \frac{k_\beta}{\sqrt{k^2 - k_\gamma^2}} \mathrm{d}k_\beta = \delta_{ll'}$$

we obtain the inverse expansion

$$\Phi^{l\beta\gamma}_{kk_{\beta\gamma}}(r_{\beta\gamma},\theta) = \int_0^{k_{\beta\gamma}} W^{l'\beta\gamma *}_{kk_{\beta\gamma}k_\beta k_\gamma} \Phi^{\beta\gamma}_{kk_\beta k_\gamma}(r_\beta, r_\gamma) \frac{k_\beta}{\sqrt{k^2 - k_\gamma^2}} \, \mathrm{d}k_\beta. \tag{4.8}$$

Putting the functions (4.3), (4.4) and the interbase coefficients (4.5) into the expansions (4.7) and (4.8), we obtain

$$\begin{aligned} J_{\beta+\frac{1}{2}S_\beta}(z\cos\theta\cos\phi) J_{\gamma+\frac{1}{2}S_\gamma}(z\sin\theta\sin\phi) &= (\sin\phi\sin\theta)^{\gamma+\frac{1}{2}S_\gamma}(\cos\phi\cos\theta)^{\beta+\frac{1}{2}S_\beta} \\ &\times \sum_{l=\beta+\gamma,\beta+\gamma+2,\ldots}^{\infty} (-1)^{\frac{1}{2}(l-\beta-\gamma)} \frac{(2l+S_\beta+S_\gamma+2)}{z} \\ &\times \frac{\Gamma(\frac{1}{2}(l+S_\beta+S_\gamma+\beta+\gamma)+1)(\frac{1}{2}(l-\beta-\gamma)+1)!}{\Gamma(\frac{1}{2}(l-\beta+\gamma+S_\gamma)+1)\Gamma(\frac{1}{2}(l+\beta-\gamma+S_\beta)+1)} \\ &\times P^{(\gamma+\frac{1}{2}S_\beta,\beta+\frac{1}{2}S_\beta)}_{\frac{1}{2}(l-\beta-\gamma)}(\cos 2\phi) P^{(\gamma+\frac{1}{2}S_\gamma,\beta+\frac{1}{2}S_\beta)}_{\frac{1}{2}(l-\beta-\gamma)}(\cos 2\theta) J_{l+\frac{1}{2}(S_\beta+S_\gamma)+1}(z) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\frac{(-1)^{\frac{1}{2}(l-\beta-\gamma)}}{z} J_{l+(S_\beta+S_\gamma)/2+1}(z)(\sin\theta)^{\gamma+\frac{1}{2}S_\gamma}(\cos\theta)^{\beta+\frac{1}{2}S_\beta} P^{(\gamma+\frac{1}{2}S_\gamma,\beta+\frac{1}{2}S_\beta)}_{\frac{1}{2}(l-\beta-\gamma)}(\cos 2\theta) \\ &\quad = \int_0^{\pi/2} (\sin\phi)^{\gamma+\frac{1}{2}S_\gamma+\frac{1}{2}} J_{\beta+\frac{1}{2}S_\beta}(z\cos\theta\cos\phi) \\ &\quad \times J_{\gamma+\frac{1}{2}S_\gamma}(z\sin\theta\sin\phi)(\cos\phi)^{\beta+\frac{1}{2}S_\beta+\frac{1}{2}} P^{(\gamma+\frac{1}{2}S_\gamma,\beta+\frac{1}{2}S_\beta)}_{\frac{1}{2}(l-\beta-\gamma)}(\cos 2\phi)\,\mathrm{d}\phi \end{aligned} \tag{4.10}$$

where $z \equiv k_{\beta\gamma} r_{\beta\gamma}$ and

$$r_\beta = r_{\beta\gamma}\cos\theta \qquad r_\gamma = r_{\beta\gamma}\sin\theta$$

$$k_\beta = k_{\beta\gamma}\cos\phi \qquad k_\gamma = k_{\beta\gamma}\sin\phi.$$

The previous two expansions are equivalent to well known formulae in the theory of Bessel functions [19], namely expansions of the product of two Bessel functions in terms of the product of one Bessel function and two Jacobi polynomials, and vice versa.

The entire procedure of contraction is illustrated in figure 5. The vertical arrows correspond to the contraction (4.2) from the S_n trees to the E_n clusters. The first tree in figure 5(*a*) contracts to bihyperspherical coordinates and the second to hyperspherical ones. The contraction of the coefficients T or overlap functions is given by equation (4.5). This is an asymptotic formula for the Racah coefficients, where the three momenta satisfy $J, m, \alpha \to \infty$. The interbase expansion (4.7) and its inverse integral expansion (4.8) between two E_n cluster diagrams (see figure 5(*b*)) are obtained from the interbase expansions (4.1) for the S_n tree diagrams. We thus obtain relations between the bihyperspherical and hyperspherical bases for the Helmholtz equation on E_n.

The contraction limit $R \to \infty$ of the interbase expansion corresponding to figure 3(*b*) with α an open end (or $q^2 = 0$; see, e.g., (4.2)) can be obtained from equations (4.7) and (4.8) by performing the substitutions $r = r_{\beta\gamma}$ and $k = k_{\beta\gamma}$.

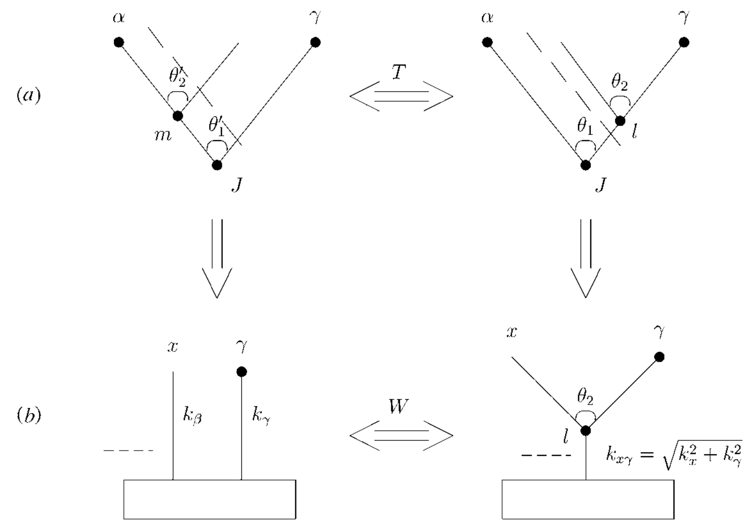

Figure 6. Contractions for two closed ends (the open one is transplanted).

4.2. *Contraction for two closed ends; open one in the middle*

The two trees in figure 6 correspond to two subgroup reductions: $O(n+1) \supset O(n_\alpha+1) \otimes O(n_\gamma)$ on the left-hand side and $O(n+1) \supset O(n_\alpha) \otimes O(n_\gamma+1)$ on the right. The overlap functions are again expressed in terms of Racah coefficients (see equation (A.3)). The corresponding interbase expansion is

$$\Psi^{\alpha\gamma}_{Jm}(\theta'_1, \theta'_2) = \sum_{l=\gamma,\gamma+1}^{J-\alpha} T^{\alpha\gamma}_{Jlm} \Psi^{\alpha\gamma}_{Jl}(\theta_1, \theta_2) \tag{4.11}$$

where the T-coefficients are given by equation (A.3) and the hyperspherical functions Ψ on both sides of equation (4.11) can be written with the help of the formulae of section 2. Since the quantum numbers $J - m - \gamma$ and $J - l - \alpha$ are even, $l - \gamma + m - \alpha$ is also even, and in the expansion (4.11) we have $l = \gamma, \gamma+2, \ldots J-\alpha$ for $(m-\alpha)$ even, and $l = \gamma+1, \gamma+3, \ldots J-\alpha$ for $(m - \alpha)$ odd.

As in the previous case, the contraction will involve three quantum numbers: J, m and α. In the contraction limit $R \to \infty$,

$$J \sim kR \qquad m \sim pR \qquad \alpha \sim qR$$
$$\theta'_1 \sim \frac{r_\gamma}{R} \qquad \theta'_2 \sim \frac{x}{R} \qquad \theta_1 \sim \frac{\sqrt{r_\gamma^2 + x^2}}{R} \tag{4.12}$$

where $k_\gamma^2 = k^2 - p^2$, $k_x^2 = p^2 - q^2$, $k_{x\gamma}^2 = k_x^2 + k_\gamma^2$, we obtain

$$\lim_{R\to\infty} (-1)^{\frac{1}{2}(m-\alpha)} \Psi^{\alpha\gamma}_{Jm}(\theta'_1, \theta'_2) = \Phi^{\gamma}_{kk_xk_\gamma}(r_\gamma, x) = \sqrt{\frac{k}{\pi k_x}} \frac{2(k^2 - k_\gamma^2)^{1/4}}{r_\gamma^{S_\gamma/2}}$$
$$\times J_{\gamma+\frac{1}{2}S_\gamma}(k_\gamma r_\gamma) \begin{cases} \cos k_x x & (m-\alpha) \text{ even} \\ -\mathrm{i}\sin k_x x & (m-\alpha) \text{ odd} \end{cases} \tag{4.13}$$
$$\lim_{R\to\infty} R^{-\frac{1}{2}(S_\gamma+1)} \Psi^{\alpha\gamma}_{Jl}(\theta_1, \theta_2) = \Phi^{l\gamma}_{kk_{x\gamma}}(\sqrt{r_\gamma^2 + x^2}, \theta_2)$$

$$= \sqrt{\frac{k(2l+S_\gamma+1)(l-\gamma)!}{\pi(l+S_\gamma+\gamma)!}} \frac{\Gamma(\gamma+\frac{1}{2}(S_\gamma+1))}{(r_\gamma^2+x^2)^{\frac{1}{4}(S_\gamma+1)}}$$

$$\times 2^{\gamma+\frac{1}{2}S_\gamma}(\sin\theta_2)^\gamma J_{l+\frac{1}{2}(S_\gamma+1)}\left(k_{x\gamma}\sqrt{r_\gamma^2+x^2}\right) P_{l-\gamma}^{(\gamma+\frac{1}{2}S_\gamma,\gamma+\frac{1}{2}S_\gamma)}(\cos\theta_2). \tag{4.14}$$

For the contractions of the interbase coefficients T, we obtain

$$\lim_{R\to\infty}(-1)^{\frac{1}{2}(m-\alpha)} T_{Jlm}^{\alpha\gamma} = W_{kk_{x\gamma}k_xk_\gamma}^{l\gamma} = \sqrt{\frac{2(2l+S_\gamma+1)}{\pi k_{x\gamma}k_x}}(k^2-k_\gamma^2)^{1/4}\left(\frac{k_\gamma}{k_{x\gamma}}\right)^{\gamma+\frac{1}{2}S_\gamma}$$

$$\times\begin{cases} \mathcal{A}^{1/2}\cdot {}_2F_1\left(-\frac{1}{2}(l-\gamma),\frac{1}{2}(l+\gamma+S_\gamma+1);\frac{1}{2};\frac{k_x^2}{k_{x\gamma}^2}\right) & (l-\gamma)\ \text{even} \\ -2\mathrm{i}\mathcal{A}^{-1/2}\left(\frac{k_x}{k_{x\gamma}}\right)\cdot {}_2F_1\left(-\frac{1}{2}(l-\gamma-1),\frac{1}{2}(l+\gamma+S_\gamma)+1;\frac{3}{2};\frac{k_x^2}{k_{x\gamma}^2}\right) & (l-\gamma)\ \text{odd} \end{cases} \tag{4.15}$$

where

$$\mathcal{A} = \frac{\Gamma\left(\frac{1}{2}(l+\gamma+S_\gamma+1)\right)\Gamma\left(\frac{1}{2}(l-\gamma+1)\right)}{\Gamma(\frac{1}{2}(l+\gamma+S_\gamma)+1)\Gamma(\frac{1}{2}(l-\gamma)+1)}.$$

Using the connection between hypergeometrical functions ${}_2F_1$ and the Gegenbauer polynomials [19], we obtain

$$W_{kk_{x\gamma}k_xk_\gamma}^{l\gamma} = \frac{(-1)^{\frac{1}{2}(l-\gamma)}2^{\gamma+\frac{1}{2}(S_\gamma+1)}}{(k^2-k_\gamma^2)^{-\frac{1}{4}}}\Gamma\left(\gamma+\frac{1}{2}(S_\gamma+1)\right)\sqrt{\frac{(2l+S_\gamma+1)(l-\gamma)!}{\pi k_xk_\gamma(l+\gamma+S_\gamma)!}}$$

$$\times(\sin\phi)^{\gamma+\frac{1}{2}(S_\gamma+1)}C_{l-\gamma}^{\gamma+\frac{1}{2}(S_\gamma+1)}(\cos\phi) \qquad \cos\phi = \frac{k_x}{k_{x\gamma}}. \tag{4.16}$$

Multiplying the interbase expansion (4.11) by the factor $R^{-\frac{1}{2}(S_\gamma+1)}$ and taking the contraction limit $R\to\infty$, we obtain ($\theta_2\equiv\theta$)

$$\Phi_{kk_xk_\gamma}^{\gamma}(r_\gamma,x) = {\sum_{l=\gamma,\gamma+1}^{\infty}}' W_{kk_{x\gamma}k_xk_\gamma}^{l\gamma}\Phi_{kk_{x\gamma}}^{l\gamma}\left(\sqrt{r_\gamma^2+x^2},\theta\right). \tag{4.17}$$

We use the orthogonality condition for the Gegenbauer polynomials [19]

$$\int_{-k_{x\gamma}}^{k_{x\gamma}} W_{kk_{x\gamma}k_xk_\gamma}^{l\gamma} W_{kk_{x\gamma}k_xk_\gamma}^{l'\gamma *}\frac{k_x\,\mathrm{d}k_x}{\sqrt{k^2-k_\gamma^2}} = 4\delta_{ll'} \tag{4.18}$$

to obtain the inverse expansion

$$\Phi_{kk_{x\gamma}}^{l\gamma}\left(\sqrt{r_\gamma^2+x^2},\theta\right) = \frac{1}{4}\int_{-k_{x\gamma}}^{k_{x\gamma}} W_{kk_{x\gamma}k_xk_\gamma}^{l\gamma *}\Phi_{kk_xk_\gamma}^{\gamma}(r_\gamma,x)\frac{k_x\,\mathrm{d}k_x}{\sqrt{k^2-k_\gamma^2}}. \tag{4.19}$$

Thus, the interbase expansion (4.11) transforms in to the expansion between the hypercylindrical and hyperspherical bases for Helmholtz equation.

Substituting formulae (4.13), (4.14) and (4.16) into expansions (4.17) and (4.19) and putting

$$k_x = k_{x\gamma}\cos\phi \qquad k_\gamma = k_{x\gamma}\sin\phi$$

$$x = \sqrt{r_\gamma^2+x^2}\cos\theta_2 \qquad r_\gamma = \sqrt{r_\gamma^2+x^2}\sin\theta_2$$

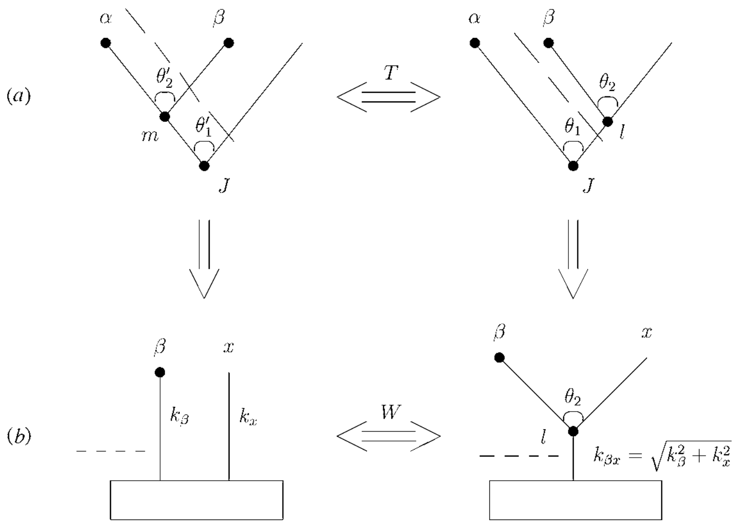

Figure 7. Contractions for two closed ends (open one on the right).

we have ($z \equiv k_{x\gamma}\sqrt{r_\gamma^2 + x^2}$)

$$J_{\gamma+\frac{1}{2}S_\gamma}(z\sin\phi\sin\theta)\begin{Bmatrix}\cos(z\cos\phi\cos\theta)\\ \sin(z\sin\phi\sin\theta)\end{Bmatrix} = \frac{2^{2\gamma+S_\gamma-1}}{\sqrt{\pi z}}\Gamma^2(\gamma+\tfrac{1}{2}(S_\gamma+1))(\sin\phi\sin\theta)^{\gamma+\frac{1}{2}S_\gamma}$$
$$\times\sum_{l=\gamma,\gamma+1}^{\infty}\frac{(2l+S_\gamma+1)(l-\gamma)!}{(l+\gamma+S_\gamma)!}$$
$$\times C_{l-\gamma}^{\gamma+\frac{1}{2}(S_\gamma+1)}(\cos\phi)C_{l-\gamma}^{\gamma+\frac{1}{2}(S_\gamma+1)}(\cos\theta)J_{l+\frac{1}{2}(S_\gamma+1)}(z) \tag{4.20}$$

and

$$J_{l+\frac{1}{2}(S_\gamma+1)}(z)(\sin\theta)^{\gamma+\frac{1}{2}S_\gamma}C_{l-\gamma}^{\gamma+\frac{1}{2}(S_\gamma+1)}(\cos\theta) = \sqrt{\frac{z}{2\pi}}\int_0^\pi \mathrm{d}\phi\,(\sin\phi)^{\gamma+\frac{1}{2}S_\gamma+1}$$
$$\times C_{l-\gamma}^{\gamma+\frac{1}{2}(S_\gamma+1)}(\cos\phi)J_{\gamma+\frac{1}{2}S_\gamma}(z\sin\phi\sin\theta)\begin{Bmatrix}\cos(z\cos\phi\cos\theta)\\ \sin(z\sin\phi\sin\theta)\end{Bmatrix}.$$

The contraction limit $R \to \infty$ of the interbase expansion corresponding to figure 3(*e*) with α an open end (or $q^2 = 0$; see, e.g., (4.12)) can be presented by formulae (4.17) and (4.19) with the substitutions $r^2 = r_\gamma^2 + x^2$ and $k^2 = k_x^2 + k_\gamma^2$.

4.3. Contraction for two closed ends, open one on the right

In this case, the left tree corresponds to the subgroup chains $O(n+1) \supset O(n_\alpha + n_\beta)$ and the right one to $O(n+1) \supset O(n_\alpha) \otimes O(n_\beta + 1)$. The overlap functions (A.4) of the appendix are again expressed in terms of the Racah coefficients. The expansion corresponding to figure 7 has the form

$$\Psi_{Jm}^{\alpha\beta}(\theta_1', \theta_2') = \sum_{l=\beta,\beta+1}^{J-\alpha} T_{Jlm}^{\alpha\beta}\Psi_{Jl}^{\alpha\beta}(\theta_1, \theta_2) \tag{4.21}$$

where the T-coefficient is given by equation (A.4) and the wavefunctions Ψ may be constructed by using the rules given in section 2. As in the previous case the quantum number l runs through

$l = \beta, \beta+2, \ldots, J-\alpha$ or $l = \beta+1, \beta+3, \ldots, J-\alpha$ depending on $J-m$ being even or odd.

In the contraction limit $R \to \infty$

$$J \sim kR \qquad m \sim pR \qquad \alpha \sim qR$$

$$\theta_1' \sim \frac{x}{R} \qquad \theta_2' \sim \frac{r_\beta}{R} \qquad \theta_1 \sim \frac{\sqrt{r_\beta^2 + x^2}}{R}$$

we obtain

$$\begin{aligned}\Phi^{\beta}_{kk_xk_\beta}(x, r_\beta) &= \lim_{R\to\infty} (-1)^{\frac{1}{2}(J-m)} \Psi^{\alpha\beta}_{Jm}(\theta_1', \theta_2') \\ &= \frac{2\sqrt{kp}}{\sqrt{\pi k_x}} (r_\beta)^{-\frac{1}{2}S_\beta} J_{\beta+\frac{1}{2}S_\beta}(k_\beta r_\beta) \begin{cases} \cos k_x x & (J-m) \text{ even} \\ -\mathrm{i} \sin k_x x & (J-m) \text{ odd} \end{cases}\end{aligned} \tag{4.22}$$

$$\begin{aligned}\Phi^{l\beta}_{kk_{x\beta}}\left(\sqrt{r_\beta^2 + x^2}, \theta_2\right) &= \lim_{R\to\infty} \Psi^{\alpha\beta}_{Jl}(\theta_1, \theta_2) \\ &= \frac{2^{\beta+\frac{1}{2}S_\beta}\Gamma(\beta + \frac{1}{2}(S_\beta+1))}{(r_\beta^2 + x^2)^{\frac{1}{4}(S_\beta+1)}} \sqrt{\frac{k(2l+S_\beta+1)(l-\beta)!}{\pi(l+\beta+S_\beta)!}} \\ &\quad \times J_{l+\frac{1}{2}(S_\beta+1)}\left(k_{x\beta}\sqrt{r_\beta^2+x^2}\right)(\cos\theta_2)^\beta C^{\beta+\frac{1}{2}(S_\beta+1)}_{l-\beta}(\sin\theta_2)\end{aligned} \tag{4.23}$$

where $k_\beta^2 = p^2 - q^2$, $k_x^2 = k^2 - p^2$ and $k_{x\beta}^2 = k_x^2 + k_\beta^2$.

For the T-coefficients we obtain

$$\begin{aligned}W^{l\beta}_{kk_{x\beta}k_xk_\beta} &= \lim_{R\to\infty} T^{\alpha\beta}_{Jlm} = \frac{1}{2^{\beta+\frac{1}{2}(S_\beta-1)}} \sqrt{\frac{(2l+S_\beta+1)(l+\beta+S_\beta)!}{(l-\beta)![\Gamma(\beta+\frac{1}{2}S_\beta+1)]^2}} \sqrt{\frac{p}{k_xk_\beta}} \\ &\quad \times \left(\frac{k_x}{k_{x\beta}}\right)^l \left(\frac{k_\beta}{k_x}\right)^\beta \left(\frac{k_\beta}{k_{x\beta}}\right)^{S_\beta/2} \\ &\quad \times {}_2F_1\left(-\tfrac{1}{2}(l-\beta), -\tfrac{1}{2}(l-\beta-1); \beta+\tfrac{1}{2}S_\beta+1; -\frac{k_\beta^2}{k_x^2}\right) \\ &= \frac{2^{\beta+\frac{1}{2}(S_\beta+1)}\Gamma(\beta+\frac{1}{2}(S_\beta+1))}{\sqrt{\pi k_x k_{x\beta}}} \left\{\frac{p(2l+S_\beta+1)(l-\beta)!}{(l+\beta+S_\beta)!}\right\}^{1/2} \\ &\quad \times (\cos\phi)^{\beta+\frac{1}{2}S_\beta} C^{\beta+\frac{1}{2}(S_\beta+1)}_{l-\beta}(\sin\phi)\end{aligned}$$

where $\cos\phi = k_\beta / k_{x\beta}$.

Multiplying the expansion (4.21) by the factor $R^{-(S_\beta+1)/2}$ and taking the contraction limit $R \to \infty$, we obtain the flat-space expansion ($\theta_2 \equiv \theta$)

$$\Phi^{\beta}_{kk_xk_\beta}(x, r_\beta) = \sum_{l=\beta,\beta+1}^{\infty} W^{l\beta}_{kk_{x\beta}k_xk_\beta} \Phi^{l\beta}_{kk_{x\beta}}\left(\sqrt{r_\beta^2+x^2}, \theta\right). \tag{4.24}$$

Using the orthogonality condition for the Gegenbauer polynomials [19], we have

$$\int_{-k_{x\beta}}^{k_{x\beta}} W^{l\beta}_{kk_{x\beta}k_xk_\beta} W^{l'\beta*}_{kk_{x\beta}k_xk_\beta} \frac{k_x\, \mathrm{d}k_x}{\sqrt{k^2 - k_\beta^2}} = 4\delta_{ll'} \tag{4.25}$$

and the inverse expansion has the following form:

$$\Phi^{l\beta}_{kk^2_{x\beta}}\left(\sqrt{r^2_\beta + x^2}, \theta\right) = \frac{1}{4}\int_{-k_{x\beta}}^{k_{x\beta}} W^{l\beta *}_{kk_{x\beta}k_x k_\beta}\, \Phi^{\beta}_{kk_x k_\beta}(r_\beta, x) \frac{k_x\, \mathrm{d}k_x}{\sqrt{k^2 - k^2_\beta}}. \tag{4.26}$$

Putting

$$k_x = k_{x\beta}\sin\phi \qquad k_y = k_{x\beta}\cos\phi$$

$$x = \sqrt{r^2_\beta + x^2}\sin\theta_2 \qquad r_\beta = \sqrt{r^2_\beta + x^2}\cos\theta_2$$

we finally obtain the expansion of the product of Bessel and trigonometric functions in terms of the product of two Gegenbauer polynomials and one Bessel function

$$\sqrt{z}J_{\beta+\frac{1}{2}S_\beta}(z\cos\theta\cos\phi)\begin{Bmatrix}\cos(z\sin\theta\sin\phi)\\ \sin(z\sin\theta\sin\phi)\end{Bmatrix} = \frac{2^{2\beta+S_\beta}\Gamma^2(\beta+\frac{1}{2}(S_\beta+1))}{\pi}(\cos\theta\cos\phi)^{\beta+\frac{1}{2}S_\beta}$$
$$\times\sum_{l=\beta,\beta+1}^{\infty}\frac{(2l+S_\beta+1)(l-\beta)!}{(l+\beta+S_\beta)!}C^{\beta+\frac{1}{2}(S_\beta+1)}_{l-\beta}(\sin\phi)C^{\beta+\frac{1}{2}(S_{\beta+1})}_{l-\beta}(\sin\theta_2)J_{l+\frac{1}{2}(S_\beta+1)}(z) \tag{4.27}$$

where ($z \equiv k_{x\beta}\sqrt{r^2_\beta + x^2}$) and the top line on the left-hand side corresponds to a summation over $l = \beta, \beta{+}2, \ldots, J-\alpha$ and the bottom one to a summation over $l = \beta{+}1, \beta{+}3, \ldots, J-\alpha$. The expansion (4.27) is related to (4.20) by the substitutions $\theta \to \pi/2-\theta$, $\phi \to \pi/2-\phi$ and $\gamma \to \beta$, $S_\gamma \to S_\beta$.

Note that the contraction limit $R \to \infty$ in the interbase expansion, corresponding to figure 3(*f*) with α an open end (or $q^2 = 0$), can be obtained from the expansion (4.24) and (4.26) by the substitutions $k^2 = k^2_{x\beta}$ and $r^2 = r^2_\beta + x^2$.

4.4. Contractions for one closed end

The tree on the left-hand side of figure 8(*a*) corresponds to the subgroup chains $O(n+1) \supset O(n_\alpha+1) \supset O(n_\alpha)$ and the right one to $O(n+1) \supset O(n_\alpha) \otimes O(2)$. The overlap functions

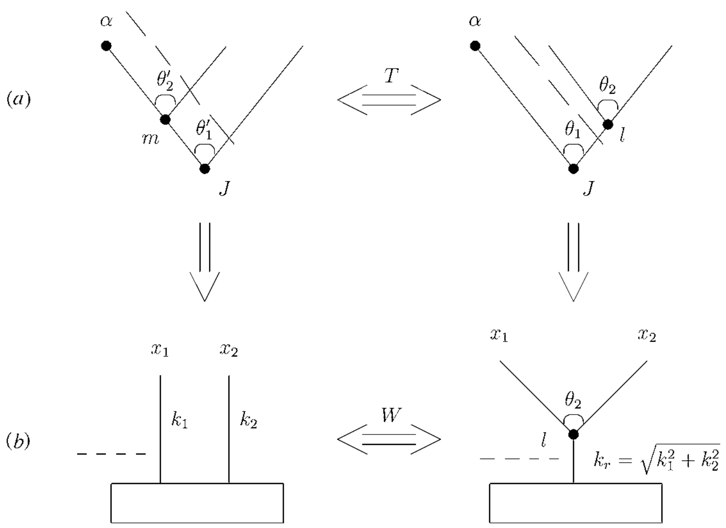

Figure 8. Contractions for one closed end.

(A.7) are expressed in terms of the Clebsch–Gordan coefficients of the $SU(2)$ group and the interbase expansion is

$$\Psi^{\alpha}_{Jm}(\theta_1', \theta_2') = \sum_{l=-(J-\alpha)}^{(J-\alpha)} (-\mathrm{i})^{m-\alpha+l}(-1)^{\frac{1}{2}(|l|-l)} \times C^{m+\frac{1}{2}S_\alpha,\alpha+\frac{1}{2}S_\alpha}_{\frac{1}{2}J+\frac{1}{4}(S_\alpha),\frac{1}{2}(\alpha+l)+\frac{1}{4}(S_\alpha);\frac{1}{2}J+\frac{1}{4}(S_\alpha),\frac{1}{2}(\alpha-l)+\frac{1}{4}(S_\alpha)} \Psi^{\alpha}_{Jl}(\theta_1, \theta_2) \tag{4.28}$$

where the wavefunctions Ψ can be written out by using the formulae in section 2, and l has the same parity as $(J-\alpha)$.

In the contraction limit $R \to \infty$

$$J \sim kR \qquad m \sim pR \qquad \alpha \sim qR \qquad \theta_1 \sim \frac{r}{R} \qquad \theta_2' \sim \frac{x_1}{R} \qquad \theta_1' \sim \frac{x_2}{R}$$

where $k_1^2 = p^2 - q^2$, $k_2^2 = k^2 - p^2$, $k_r^2 = k_1^2 + k_2^2$, we obtain ($\theta \equiv \theta_2$)

$$\lim_{R\to\infty} \frac{1}{\sqrt{R}} \Psi^{\alpha}_{Jl}(\theta_1, \theta_2) = \Phi^{l}_{kk_r}(r, \theta) = \sqrt{\frac{k}{\pi}} J_{|l|}(k_r r)\, \mathrm{e}^{\mathrm{i}l\theta} \tag{4.29}$$

$$\lim_{R\to\infty} (-1)^{\frac{1}{2}(J-\alpha)} \Psi^{\alpha}_{Jm}(\theta_1', \theta_2') = \Phi_{kk_1k_2}(x_1, x_2) = \sqrt{\frac{4pk}{\pi^2 k_1 k_2}} \times \begin{cases} \cos(k_1x_1)\cos(k_2x_2) & (J-m) \text{ even}, (m-\alpha) \text{ even} \\ -\mathrm{i}\cos(k_1x_1)\sin(k_2x_2) & (J-m) \text{ odd}, (m-\alpha) \text{ even} \\ -\mathrm{i}\sin(k_1x_1)\cos(k_2x_2) & (J-m) \text{ even}, (m-\alpha) \text{ odd} \\ -\sin(k_1x_1)\sin(k_2x_2) & (J-m) \text{ odd}, (m-\alpha) \text{ odd} \end{cases} \tag{4.30}$$

$$\begin{aligned} &\lim_{R\to\infty} (-1)^{-\frac{1}{2}(J-\alpha)} \sqrt{R} (-\mathrm{i})^{m-\alpha+l}(-1)^{\frac{1}{2}(|l|-l)} C^{m+\frac{1}{2}S_\alpha,\alpha+\frac{1}{2}S_\alpha}_{\frac{1}{2}J+\frac{1}{4}(S_\alpha),\frac{1}{2}(\alpha+l)+\frac{1}{4}(S_\alpha);\frac{1}{2}J+\frac{1}{4}(S_\alpha),\frac{1}{2}(\alpha-l)+\frac{1}{4}(S_\alpha)} = W^{l}_{kk_1k_2} \\ &\quad = (\mathrm{i})^l (-1)^{\frac{1}{2}(|l|-l)} \sqrt{\frac{4p}{\pi k_1 k_2}} \times \begin{cases} {}_2F_1\left(-\frac{1}{2}(|l|), \frac{1}{2}(|l|); \frac{1}{2}; -\frac{k_2^2}{k_r^2}\right) & (J-m) \text{ even} \\ -\mathrm{i}l\frac{k_2}{k_r}\,{}_2F_1\left(\frac{1}{2}(1-|l|), \frac{1}{2}(1+|l|); \frac{3}{2}; -\frac{k_2^2}{k_r^2}\right) & (J-m) \text{ odd} \end{cases} \\ &\quad = (\mathrm{i})^l (-1)^{\frac{1}{2}(|l|-l)} \sqrt{\frac{4pk_r}{\pi k_1 k_2}} \begin{cases} \cos|l|\phi & (J-m) \text{ even} \\ -\mathrm{i}\sin|l|\phi & (J-m) \text{ odd}, \cos\phi = k_1/k_r. \end{cases} \end{aligned} \tag{4.31}$$

Multiplying the expansion (4.28) by the factor $(-1)^{\frac{1}{2}(J-\alpha)}$ and taking the contraction limit $R \to \infty$, we obtain

$$\mathrm{e}^{-\mathrm{i}k_1x_1} \begin{Bmatrix} \cos(k_2x_2) \\ \sin(k_2x_2) \end{Bmatrix} = \sum_{l=-\infty}^{\infty} (\mathrm{i})^l (-1)^{\frac{1}{2}(|l|-l)} \sqrt{k_r} \begin{Bmatrix} \cos|l|\phi \\ \sin|l|\phi \end{Bmatrix} J_{|l|}(k_r r)\, \mathrm{e}^{\mathrm{i}l\theta} \tag{4.32}$$

where $r = \sqrt{x_1^2 + x_2^2}$ and $\tan\theta = x_2/x_1$. The inverse expansion is

$$J_{|l|}(k_r r)\, \mathrm{e}^{\mathrm{i}l\theta} = \frac{(\mathrm{i})^l(-1)^{\frac{1}{2}(|l|-l)}}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}l\phi - \mathrm{i}k_r r\cos(\theta-\phi)}\, \mathrm{d}\phi. \tag{4.33}$$

For $\theta = 0$ the latter formula is equivalent to a well known formula in the theory of Bessel functions [19].

Let us sum up. We explicitly gave the contractions of T-coefficients corresponding to figures 3(a), (c), (d) and (g), in sections 4.1–4.4, respectively. Those for figures 3(b), (e), (f) and (h) are obtained as the corresponding limits for $q = 0$. In each case, graphically a closed end on the left is replaced by an open one in the limit. We recall that this left end is 'cut-off' in the contraction.

5. The group $O(3)$

The general formulae of section 4 simplify considerably for $n = 2$ and 3. We give some specific results here, as illustrations of the general ones.

For the sphere S_2 all three ends of the tree diagrams are open, as in figure 3(h). Two types of three diagrams exist for the sphere S_2, both shown in figure 9(a). In [2] we introduced subgroup diagrams for the groups $O(n+1)$ and $E(n)$, illustrating graphically possible chains of subgroups. We also gave a relation between tree diagrams for coordinates and subgroup

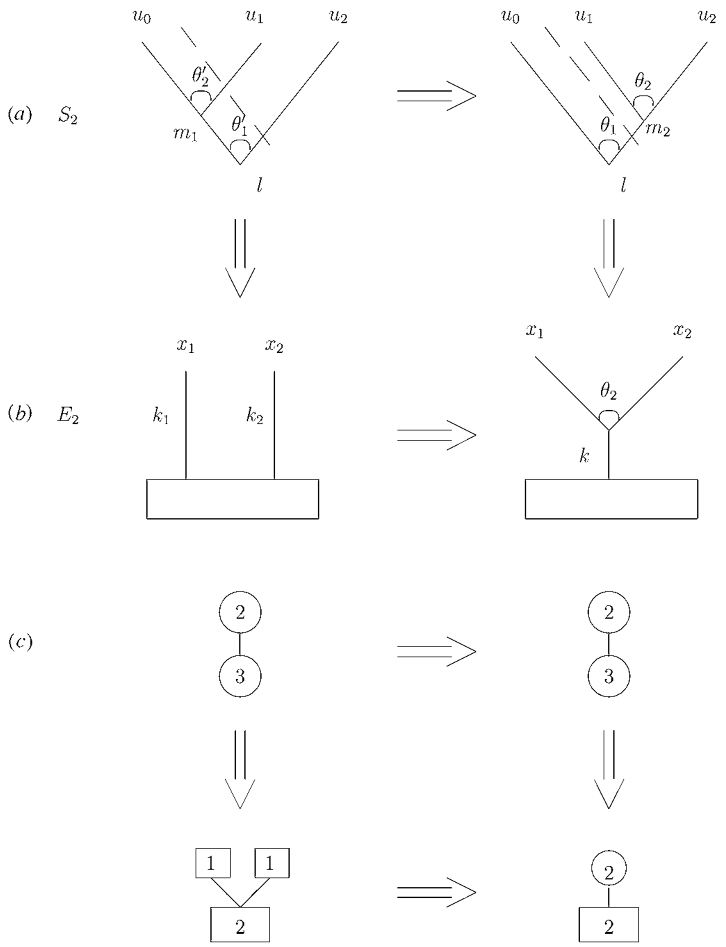

Figure 9. Tree diagrams and subgroup diagrams illustrating $S_2 \to E_2$ contractions.

diagrams. For $O(3)$ only one chain exists, namely $O(3) \supset O(2)$, for $E(2)$ two exist, namely $E(2) \supset O(2)$ and $E(2) \supset E(1) \otimes E(1)$. All are shown in figure 9(c). Circles denote $O(n)$ groups, squares $E(n)$ ones. The value of n is given in the circle, or square, respectively. The two diagrams in figure 9(a) both correspond to $O(3) \supset O(2)$ chains, but one privileges the (01) pair, and for the other the (12) one.

The spherical functions corresponding to the two trees are connected by the interbase expansion

$$Y_{lm_1}\left(\tfrac{1}{2}\pi - \theta_1', \theta_2'\right) = \sum_{m_2=-l}^{l} D^l_{m_2,m_1}(\tfrac{1}{2}\pi, \tfrac{1}{2}\pi, 0) Y_{lm_2}(\theta_1, \theta_2) \tag{5.1}$$

so that the overlap functions are the Wigner rotation matrices $D^l_{m_2,m_1}(\alpha, \beta, \gamma) = \mathrm{e}^{-\mathrm{i}m_2\alpha} d^l_{m_2,m_1}(\beta)\, \mathrm{e}^{-\mathrm{i}m_1\gamma}$ [7, 23, 24]. The angles in both sides of the expansions are connected by the relations

$$\begin{aligned} u_0 &= R\cos\theta_1 = R\cos\theta_1'\cos\theta_2' \\ u_1 &= R\sin\theta_1\cos\theta_2 = R\cos\theta_1'\sin\theta_2' \\ u_2 &= R\sin\theta_1\sin\theta_2 = R\sin\theta_1'. \end{aligned}$$

The expansion (5.1) corresponds to an ‘elementary’ transformation of the $O(3)$ tree diagram in figure 9: the branch leading to the Cartesian coordinate u_1 is ‘transplanted’ from the u_0 branch to the u_2 one.

Let us now consider the contraction $R \to \infty$ for the interbasis expansion (5.1). Contractions of basis functions were presented earlier [1, 2]. In order to obtain the corresponding limits of the Wigner D-functions, we use an integral representation for the function $d^l_{m_2,m_1}(\pi/2)$

$$d^l_{m_2,m_1}(\tfrac{1}{2}\pi) = (-1)^{\frac{1}{2}(l-m_1)} \frac{2^l}{\pi} \left\{ \frac{(l+m_2)!(l-m_2)!}{(l+m_1)!(l-m_1)!} \right\}^{1/2} \int_0^{\pi} (\sin\alpha)^{l-m_1} (\cos\alpha)^{l+m_1} \mathrm{e}^{2\mathrm{i}m_2\alpha}\, \mathrm{d}\alpha$$

and the formulae [19]

$$\cos(2n\alpha) = T_n(\cos 2\alpha) \qquad \sin(2n\alpha) = \sin 2\alpha \cdot U_{n-1}(\cos 2\alpha)$$

where $T_l(x)$ and $U_l(x)$ are Tchebyshev polynomials of the first and second kind. After integrating over α, we obtain a representation of the Wigner D-functions in terms of the hypergeometrical function ${}_3F_2$ (of argument 1):

$$D^l_{m_2,m_1}(\tfrac{1}{2}\pi, \tfrac{1}{2}\pi, 0) = \frac{(-1)^{\frac{1}{2}(l+m_2-m_1)}}{\sqrt{\pi}\, l!} \sqrt{(l+m_2)!(l-m_2)!}$$
$$\times \begin{cases} \left\{ \dfrac{\Gamma\left(\frac{1}{2}(l+m_1+1)\right)\Gamma\left(\frac{1}{2}(l-m_1+1)\right)}{\Gamma\left(\frac{1}{2}(l+m_1)+1\right)\Gamma\left(\frac{1}{2}(l-m_1)+1\right)} \right\}^{1/2} & \\ \quad \times {}_3F_2\left(\left. \begin{matrix} -m_2, m_2, \frac{1}{2}(l+m_1+1) \\ \frac{1}{2}, l+1 \end{matrix} \right| 1 \right) & (l-m_1) \text{ even} \\ \dfrac{2\mathrm{i}l}{(l+1)} \left\{ \dfrac{\Gamma\left(\frac{1}{2}(l+m_1)+1\right)\Gamma\left(\frac{1}{2}(l-m_1)+1\right)}{\Gamma\left(\frac{1}{2}(l+m_1+1)\right)\Gamma\left(\frac{1}{2}(l-m_1+1)\right)} \right\}^{1/2} & \\ \quad \times {}_3F_2\left(\left. \begin{matrix} -m_2+1, m_2+1, \frac{1}{2}(l+m_1)+1 \\ \frac{3}{2}, l+2 \end{matrix} \right| 1 \right) & (l-m_1) \text{ odd.} \end{cases} \tag{5.2}$$

Consider now the contraction limit $R \to \infty$ in the expansion (5.1). For large R we put

$$\begin{aligned} &l \sim kR \qquad m_1 \sim k_1 R \qquad \theta_1 \sim \frac{r}{R} \\ &\theta_1' \sim \frac{y}{R} \qquad \theta_2' \sim \frac{x}{R} \qquad R \to \infty \end{aligned} \tag{5.3}$$

where $k^2 = k_1^2 + k_2^2$, and have [2]

$$\lim_{R\to\infty} \frac{1}{\sqrt{R}} Y_{lm_2}(\theta_1, \theta_2) = (-1)^{\frac{1}{2}(m_2+|m_2|)} \sqrt{k} J_{|m_2|}(kr) \frac{e^{im_2\theta_2}}{\sqrt{2\pi}} \tag{5.4}$$

$$\lim_{R\to\infty} (-1)^{-\frac{1}{2}(l-|m_1|)} Y_{lm_1}\left(\tfrac{1}{2}\pi - \theta_1', \theta_2'\right) = \sqrt{\frac{k}{k_2}} \frac{e^{ik_1x}}{\pi} \begin{cases} \cos k_2 y & (l - |m_1|) \text{ even} \\ -i \sin k_2 y & (l - |m_1|) \text{ odd.} \end{cases} \tag{5.5}$$

Using known asymptotic formulae [19] for the ${}_3F_2$ functions and Γ functions in equation (5.2) we obtain

$$\begin{aligned} \lim_{R\to\infty} (-1)^{-\frac{1}{2}(l-|m_1|)} \sqrt{R} D^l_{m_2,m_1}\left(\tfrac{1}{2}\pi, \tfrac{1}{2}\pi, 0\right) &= (-1)^{\frac{1}{2}m_2} \sqrt{\frac{2}{\pi k}} \\ &\times \begin{cases} \left(\dfrac{k^2}{k_2}\right)^{1/4} {}_2F_1\left(-m_2, m_2; \dfrac{1}{2}; \dfrac{k+k_1}{2k}\right) & (l - m_1) \text{ even} \\ -i m_2 \left(\dfrac{k_2}{k^2}\right)^{1/4} {}_2F_1\left(-m_2+1, m_2+1; \dfrac{3}{2}; \dfrac{k+k_1}{2k}\right) & (l - m_1) \text{ odd} \end{cases} \\ &= (-1)^{\frac{3}{2}m_2} \sqrt{\frac{2}{\pi k_2}} \begin{cases} \cos m_2\varphi & (l - m_1) \text{ even} \\ i \sin m_2 \varphi & (l - m_1) \text{ odd} \end{cases} \end{aligned} \tag{5.6}$$

where $\cos\varphi = k_1/k$.

Multiplying the interbase expansion (5.1) by the factor $(-1)^{-\frac{1}{2}(l-|m_1|)}$ and taking the contraction limit $R \to \infty$ we obtain ($\theta \equiv \theta_2, m \equiv m_2$)

$$e^{ik_1x} \begin{Bmatrix} \cos k_2 y \\ \sin k_2 y \end{Bmatrix} = \sum_{m=-\infty}^{\infty} (i)^{|m|} \begin{Bmatrix} \cos m\varphi \\ -\sin m\varphi \end{Bmatrix} J_{|m|}(kr)\, e^{im\theta} \tag{5.7}$$

or in exponential form

$$e^{ikr\cos(\theta-\varphi)} = \sum_{m=-\infty}^{\infty} (i)^m J_m(kr)\, e^{im(\theta-\varphi)}. \tag{5.8}$$

The inverse expansion is

$$J_m(kr)\, e^{im\theta} = \frac{(-i)^m}{2\pi} \int_0^{2\pi} e^{im\varphi - ikr\cos(\theta-\varphi)}\, d\varphi. \tag{5.9}$$

For $\theta = 0$ the latter two formulae are equivalent to well known formulae in the theory of Bessel functions [19], namely expansions of plane waves in terms of cylindrical ones and vice versa.

The entire procedure is illustrated in figure 9. The vertical arrows correspond to the contraction (5.3). The $O(3)$ interbasis expansion (5.1) has contracted to the E(2) interbasis expansion (5.8) and its inverse (5.9), i.e. the relations between plane and spherical waves. The contraction of the overlap functions is given by equation (5.6): an asymptotic formula for Wigner D-functions.

We recall [2] that the E_n 'cluster' diagrams are obtained from the S_n tree diagrams by cutting along the dotted lines in figures 4 and 9. The dotted line becomes the basis for the E_n (in this case E_2) diagram. Thus two topologically equivalent tree diagrams go into inequivalent cluster diagrams. The first contracts to Cartesian coordinates, the second to polar ones. In terms of subgroup diagrams the situation is illustrated in figure 9(c). Equations (5.8) and (5.9) are special limiting cases of equations (4.32) and (4.33), respectively.

6. The group $O(4)$

Five types of tree diagrams exist for the sphere $S_3 \sim O(4)/O(3)$. Two of them are shown in figure 10(a), two more in figure 11(a), the fifth on the left-hand side of figure 12(a). Elementary transitions correspond to the transplanting of a branch (or a twig) to a neighbouring branch. All transitions can be composed from elementary ones.

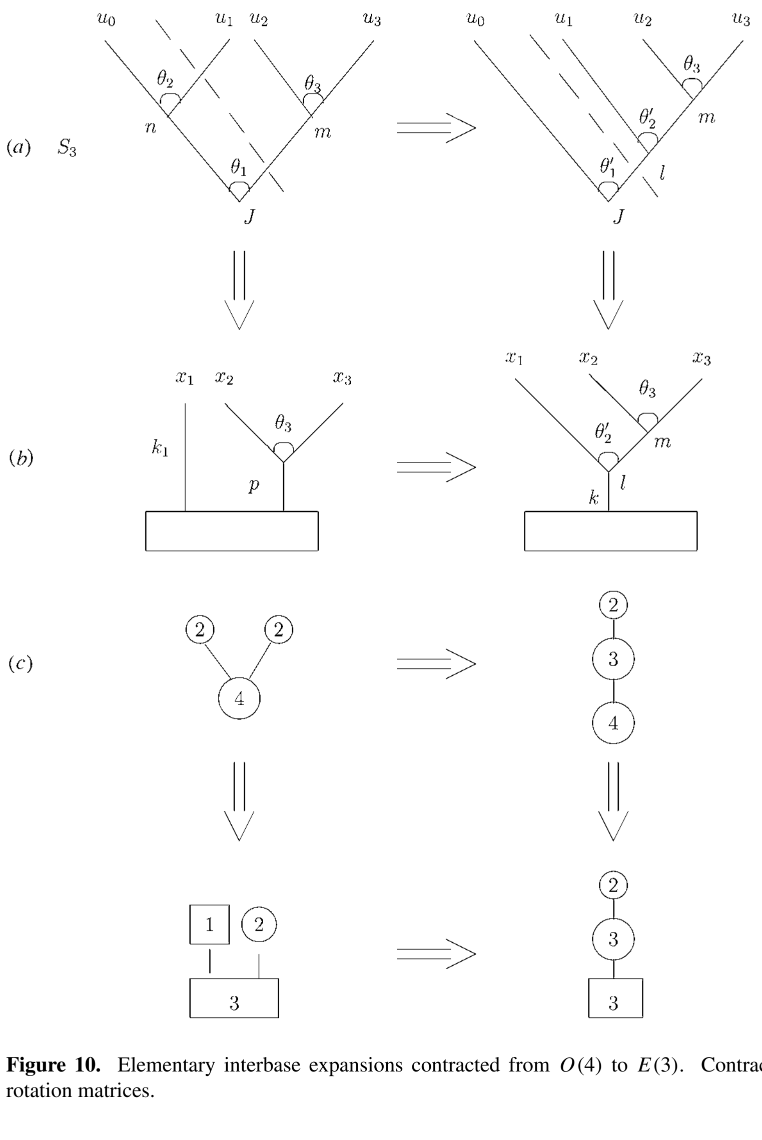

Figure 10. Elementary interbase expansions contracted from $O(4)$ to $E(3)$. Contractions of rotation matrices.

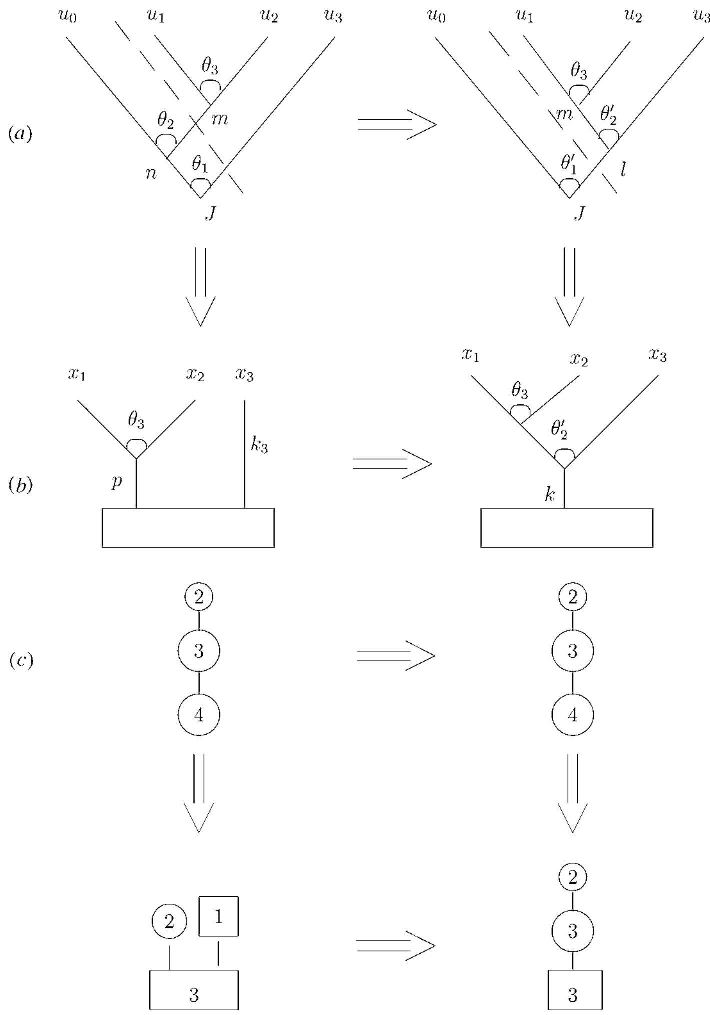

Figure 11. Elementary interbase expansions contracted from $O(4)$ to $E(3)$. Contractions of Racah coefficients.

Elementary transitions for $O(4)$ can involve either three open ends, or one closed and two open ones. For three open ends we obtain the same results as for $O(3)$. For a closed end we obtain a special cases of the formulae corresponding to figures 3(*e*), (*f*) or (*g*).

6.1. Contractions of Clebsch–Gordan coefficients

The tree on the left-hand side of figure 10(*a*) corresponds to the subgroup chain $O(4) \supset O(2) \otimes O(2)$, as indicated in figure 10(*c*). The one on the right-hand side corresponds to the chain $O(4) \supset O(3) \supset O(2)$.

The translated branch is an open one and we are in the case of figure 3(*e*).

The interbase expansions no longer correspond to a rotation of the sphere, but to a recoupling of some of the angular momenta involved. The overlap functions are expressed in

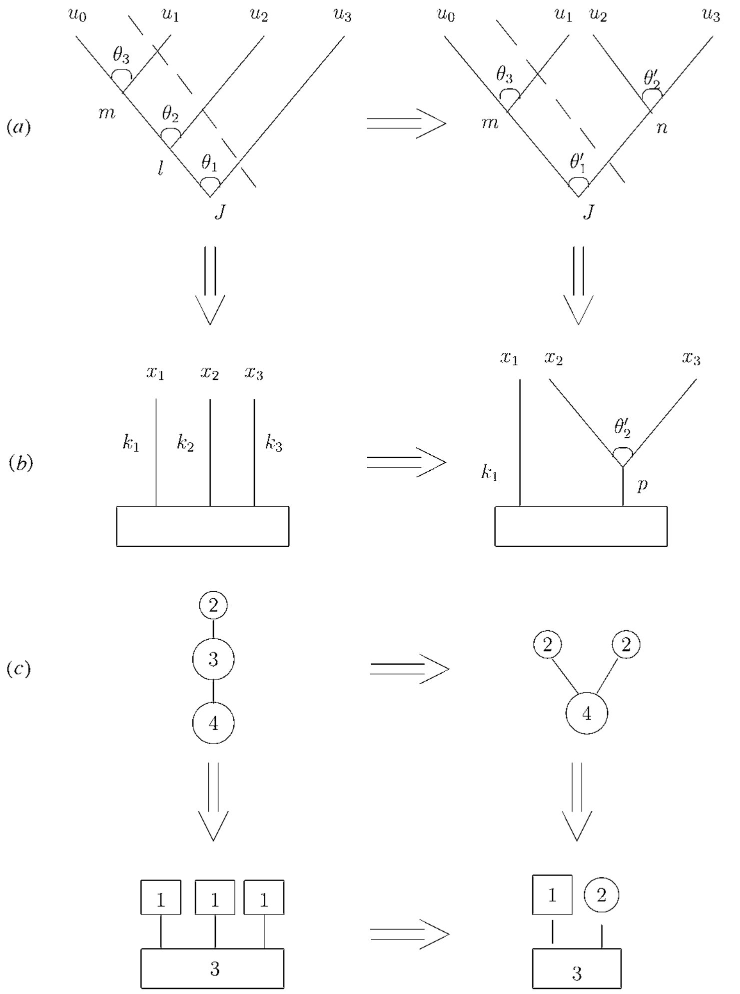

Figure 12. Elementary interbase expansions contracted from $O(4)$ to $E(3)$. Contractions of Clebsch–Gordan coefficients.

terms of Clebsch–Gordan coefficients of the $O(3)$ group and we have

$$\Psi_{Jnm}(\theta_1, \theta_2, \theta_3) = \sum_{l=|m|}^{J} (\mathrm{i})^{l-|m|}(-1)^{\frac{1}{2}(J-|m|-n)} C^{l,|m|}_{\frac{1}{2}J, \frac{1}{2}(|m|+n); \frac{1}{2}J, \frac{1}{2}(|m|-n)} \Psi_{Jlm}(\theta'_1, \theta'_2, \theta_3) \tag{6.1}$$

where

$$\begin{aligned} u_0 &= R\cos\theta_1\cos\theta_2 = R\cos\theta'_1 \\ u_1 &= R\cos\theta_1\sin\theta_2 = R\sin\theta'_1\cos\theta'_2 \\ u_2 &= R\sin\theta_1\cos\theta_3 = R\sin\theta'_1\sin\theta'_2\cos\theta_3 \\ u_3 &= R\sin\theta_1\sin\theta_3 = R\sin\theta'_1\sin\theta'_2\sin\theta_3 \end{aligned} \tag{6.2}$$

and $C^{l,\gamma}_{a,\alpha;b,\beta}$ are the Clebsch–Gordan coefficients for the $O(3)$ group. The corresponding hyperspherical functions have the form

$$\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \frac{\sqrt{2J+2}}{2\pi}\sqrt{\frac{(\frac{1}{2}(J+|m|+|n|))!(\frac{1}{2}(J-|m|-|n|))!}{(\frac{1}{2}(J+|m|-|n|))!(\frac{1}{2}(J-|m|+|n|))!}}\mathrm{e}^{\mathrm{i}n\theta_2}\mathrm{e}^{\mathrm{i}m\theta_3}$$
$$\times(\sin\theta_1)^{|m|}(\cos\theta_1)^{|n|}P^{(|m|,|n|)}_{\frac{1}{2}(J-|m|-|n|)}(\cos 2\theta_1) \tag{6.3}$$

$$\Psi_{Jlm}(\theta_1',\theta_2',\theta_3) = \frac{\sqrt{(2J+1)(J+l+1)!(J-l)!}}{2^{l+1}\Gamma(J+\frac{3}{2})}(\sin\theta_1')^l P^{(l+\frac{1}{2},l+\frac{1}{2})}_{J-l}(\cos\theta_1')Y_{lm}(\theta_2',\theta_3) \tag{6.4}$$

where $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials. We again make use of the Beltrami coordinates (3.4) (with $n=3$). In the contraction limit $R\to\infty$ and

$$\theta_1' \to \frac{r}{R} \qquad \theta_1 \to \frac{\rho}{R} \qquad \theta_2 \to \frac{x_1}{R} \qquad J\sim kR \qquad n\sim k_1 R$$

where $r=\sqrt{x_1^2+\rho^2}=\sqrt{x_1^2+x_2^2+x_3^2}$, $k=\sqrt{k_1^2+p^2}=\sqrt{k_1^2+k_2^2+k_3^2}$. We have [2]

$$\lim_{R\to\infty}\frac{1}{\sqrt{R}}\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \Phi_{kk_1m}(x_1,\rho,\theta_3) = \sqrt{\frac{k}{\pi}}J_{|m|}(p\rho)\,\mathrm{e}^{\mathrm{i}k_1x_1}\frac{\mathrm{e}^{\mathrm{i}m\theta_3}}{\sqrt{2\pi}} \tag{6.5}$$

and

$$\lim_{R\to\infty}\frac{1}{R}\Psi_{Jlm}(\theta_1',\theta_2',\theta_3) = \Phi_{klm}(r,\theta_2',\theta_3) = \sqrt{\frac{k}{r}}J_{l+\frac{1}{2}}(kr)Y_{lm}(\theta_2',\theta_3). \tag{6.6}$$

We take the Clebsch–Gordan coefficients in the form

$$C^{l,|m|}_{\frac{1}{2}J,\frac{1}{2}(|m|+n);\frac{1}{2}J,\frac{1}{2}(|m|-n)} = (-1)^{\frac{1}{2}(J-|m|-n)}\frac{(J)!}{(|m|)!}\sqrt{\frac{(2l+1)(l+|m|)!}{(J-l)!(J+l+1)!(l-|m|)!}}$$
$$\times\sqrt{\frac{(\frac{1}{2}(J+|m|-|n|))!(\frac{1}{2}(J-|m|+|n|))!}{(\frac{1}{2}(J+|m|+|n|))!(\frac{1}{2}(J-|m|-|n|))!}}$$
$$\times\,{}_3F_2\left\{\begin{matrix}-\frac{1}{2}(J-n-|m|),-l,l+1\\ -J,|m|+1,\end{matrix}\middle|1\right\}. \tag{6.7}$$

In the contraction limit $R\to\infty$, we obtain

$$\lim_{R\to\infty}\sqrt{R}(-1)^{\frac{1}{2}(J-|m|-n)}C^{l,|m|}_{\frac{1}{2}J,\frac{1}{2}(|m|+|n|);\frac{1}{2}J,\frac{1}{2}(|m|-|n|)}$$
$$= W^l_{k|m|}(\cos\phi) = \sqrt{\frac{(2l+1)(l+|m|)!}{k(l-|m|)!}}\frac{(\sin\phi)^{|m|}}{2^{|m|}|m|!}$$
$$\times\,{}_2F_1\left(-l+|m|,l+|m|+1;|m|+1;\tfrac{1}{2}(1-\cos\phi)\right) = \sqrt{\frac{2}{k}}\mathcal{P}_l^{|m|}(\cos\phi) \tag{6.8}$$

where

$$\mathcal{P}_l^{|m|}(x) = \sqrt{\frac{(2l+1)(l-|m|)!}{2(l+|m|)!}}P_l^{|m|}(x)$$

are the orthonormalized Legendre polynomials and $\cos\phi = p/k$. Thus the interbase expansion (6.1) transforms into the expansion between the cylindrical and spherical bases for the Helmholtz equation

$$\Phi_{kk_1m}(x_1,\rho,\theta_3) = \sum_{l=|m|}^{\infty}W^l_{k|m|}(\cos\phi)\Phi_{klm}(r,\theta_2',\theta_3). \tag{6.9}$$

We use the formula

$$\int_0^\pi W^l_{k|m|}(\cos\phi)W^{*l'}_{k|m|}(\cos\phi)\sin\phi\,\mathrm{d}\phi = 2\delta_{l,l'} \tag{6.10}$$

to obtain the inverse expansion

$$\Phi_{klm}(r,\theta_2',\theta_3) = \tfrac{1}{2}\int_0^\pi W^{*l}_{k,|m|}(\cos\phi)\Phi_{kk_1m}(x_1,\rho,\theta_3)\sin\phi\,\mathrm{d}\phi. \tag{6.11}$$

Putting the functions (6.5), (6.6) and interbase coefficients (6.8) into the expansions (6.9) and (6.11), we obtain

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}kr\cos\phi\cos\theta_2'}J_{|m|}(kr\sin\phi\sin\theta_2') &= \sum_{l=|m|}^{\infty}(\mathrm{i})^{l+m}\frac{1}{\sqrt{kr}}J_{l+\frac{1}{2}}(kr)\mathcal{P}_l^{|m|}(\cos\phi)\mathcal{P}_l^{|m|}(\cos\theta_2')\\ \frac{1}{\sqrt{kr}}J_{l+\frac{1}{2}}(kr)\mathcal{P}_l^{|m|}(\cos\theta_2')&\\ = \frac{(-\mathrm{i})^{l+m}}{\sqrt{2\pi}}\int_0^\pi \mathrm{e}^{\mathrm{i}kr\cos\phi\cos\theta_2'}&J_{|m|}(kr\sin\phi\sin\theta_2')\mathcal{P}_l^{|m|}(\cos\phi)\sin\phi\,\mathrm{d}\phi.\end{aligned} \tag{6.12}$$

The previous two expansions coincide with well known formulae in the theory of the Bessel functions [19]. They are special cases of the limits of equation (4.20) mentioned at the end of section 4.2.

6.2. Contraction of Racah coefficients

In this case both trees in figure 11(*a*) correspond to isomorphic subgroup chains $O(4) \supset O(3) \supset O(2)$. The twig leading to the Cartesian coordinates (u_1, u_2) is transplanted to the neighbouring branch, so an $O(2)$ subgroup is moved from the $O(3)$ subgroup (012) to the (123) one. The transplanted branch is a closed one, as in figure 3(*f*). In the contraction the (012) subgroup is destroyed, the (123) one survives (see the 'cut' lines in figure 11(*a*)).

The $O(4)$ interbase expansion in this case is

$$\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \sum_{l=|m|}^{J} T^l_{Jnm}\Psi_{Jlm}(\theta_1',\theta_2',\theta_3) \tag{6.13}$$

where

$$\begin{aligned}u_0 &= R\cos\theta_1\cos\theta_2 = R\cos\theta_1'\\ u_1 &= R\cos\theta_1\sin\theta_2\cos\theta_3 = R\sin\theta_1'\cos\theta_2'\cos\theta_3\\ u_2 &= R\cos\theta_1\sin\theta_2\sin\theta_3 = R\sin\theta_1'\cos\theta_2'\sin\theta_3\\ u_3 &= R\sin\theta_1 = R\sin\theta_1'\sin\theta_2'\end{aligned} \tag{6.14}$$

(see figure 11(*a*)). The hyperspherical wavefunctions corresponding to these two trees are

$$\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \frac{\sqrt{(2J+1)(J+n+1)!(J-n)!}}{2^{n+1}\Gamma(J+\frac{3}{2})}(\cos\theta_1)^n P_{J-n}^{(n+\frac{1}{2},n+\frac{1}{2})}(\sin\theta_1)Y_{nm}(\theta_2,\theta_3) \tag{6.15}$$

and

$$\Psi_{Jlm}(\theta_1',\theta_2',\theta_3) = \frac{\sqrt{(2J+1)(J+l+1)!(J-l)!}}{2^{l+1}\Gamma(J+\frac{3}{2})}(\sin\theta_1')^l P_{J-l}^{(l+\frac{1}{2},l+\frac{1}{2})}(\cos\theta_1')Y_{lm}(\tfrac{1}{2}\pi-\theta_2',\theta_3) \tag{6.16}$$

respectively. The interbase coefficients T^l_{Jnm} are expressed [7] in terms of Racah coefficients, in turn expressed in terms of the ${}_4F_3$ hypergeometric function:

$$T^l_{Jnm} = \left[\tfrac{1}{2}(1+(-1)^{J-n+l-m})\right]\sqrt{\frac{(2l+1)(2n+1)(n+|m|)!(l+|m|)!(J-l)!(J-n)!}{(n-|m|)!(l-|m|)!(J+n+1)!(J+l+1)!}}$$

$$\times(-1)^{\frac{1}{2}(J-n+l-m)}\frac{2^{l+n-2m}}{|m|!}\frac{\Gamma(\frac{1}{2}(J-n-l+|m|)+1)}{\Gamma(\frac{1}{2}(J+n+l-|m|)+1)}$$

$$\times {}_4F_3\left\{\begin{matrix}-\frac{1}{2}(n-|m|), -\frac{1}{2}(n-|m|-1), -\frac{1}{2}(l-|m|), -\frac{1}{2}(l-|m|-1)\\ |m|+1, -\frac{1}{2}(J+n+l-|m|), \frac{1}{2}(J-n-l+|m|)+1\end{matrix}\middle|1\right\}.$$

In the contraction limit $R\to\infty$ and

$$\theta_1 \sim \frac{x_3}{R} \qquad \theta_2 \sim \frac{\rho}{R} \qquad \theta_1' \sim \frac{r}{R} \qquad J\sim kR \qquad n\sim pR$$

where $r=\sqrt{\rho^2+x_3^2}$ and $k=\sqrt{p^2+k_3^2}$, we obtain [2]

$$\lim_{R\to\infty}\frac{(-1)^{-\frac{1}{2}(J-n)}}{\sqrt{R}}\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \Phi_{kpm}(\rho,x_3,\theta_3)$$

$$= \sqrt{k}J_m(p\rho)\frac{\mathrm{e}^{\mathrm{i}m\theta_3}}{\pi}\begin{cases}\cos k_3x_3 & (J-n)\ \text{even}\\ -\mathrm{i}\sin k_3x_3 & (J-n)\ \text{odd}\end{cases}$$

$$\lim_{R\to\infty}\frac{1}{R}\Psi_{Jlm}(\theta_1',\theta_2',\theta_3) = \Phi_{klm}(r,\theta_2',\theta_3) = \sqrt{\frac{k}{r}}J_{l+\frac{1}{2}}(kr)Y_{lm}(\tfrac{1}{2}\pi-\theta_2',\theta_3).$$

For the contractions of interbase coefficients T^l_{Jnm} we obtain

$$\lim_{R\to\infty}(-1)^{-\frac{1}{2}(J-n)}\sqrt{R}T^l_{Jnm} = W^l_{k|m|}(\cos\phi) = \frac{(-1)^{\frac{1}{2}(l-m)}}{|m|!}\sqrt{\frac{(2l+1)(l+|m|)!}{2k(l-|m|)!}}$$

$$\times(\cot\phi)^{|m|+\frac{1}{2}}(\sin\phi)^l{}_2F_1\left(-\tfrac{1}{2}(l-|m|), -\tfrac{1}{2}(l-|m|-1); |m|+1; -\cot^2\phi\right)$$

$$= (-1)^{\frac{1}{2}(l+|m|)}\sqrt{\frac{2}{k}}(\cot\phi)^{1/2}\mathcal{P}_l^{|m|}(\sin\phi) \tag{6.17}$$

where $\cos\phi = p/k$. The interbase expansion in equation (6.13) transforms into the expansion between the cylindrical and spherical bases for the Helmholtz equation

$$\frac{1}{\sqrt{\pi}}J_m(p\rho)\left\{\begin{matrix}\cos k_3x_3\\ -\mathrm{i}\sin k_3x_3\end{matrix}\right\} = \sum_l \frac{(-1)^{\frac{1}{2}(l-|m|)}}{\sqrt{kr}}J_{l+\frac{1}{2}}(kr)(\cot\phi)^{1/2}\mathcal{P}_l^{|m|}(\sin\phi)\mathcal{P}_l^{|m|}(\sin\theta_2') \tag{6.18}$$

where the top line on the left-hand side corresponds to a summation over $l=|m|, |m|+2, |m|+4, \ldots$ and the bottom one to a summation over $l=|m|+1, |m|+3, \ldots$ on the right-hand side. The $E(3)$ expansion (6.18) is related to the expansion (6.12) by the substitution $k_1 = k\cos\phi\to k_3$, $x_1 = r\cos\theta_2'\to x_3$, $\phi\to\pi/2-\phi$ and $\theta_2'\to\pi/2-\theta_2'$. Equation (6.18) is the limit of a special case of expansion (4.27).

6.3. Further contractions of Clebsch–Gordan coefficients

As in figure 10, the two $O(4)$ trees in figure 12 correspond to two different subgroup reductions: $O(4) \supset O(3) \supset O(2)$ on the left and $O(4) \supset O(2) \otimes O(2)$ on the right. As in the case of figure 10 we are transplanting an open end. However, the contraction is different, as can be seen by comparing figures 10(a) and 12(a) (the dotted lines are different). Since a recoupling of momenta is involved the overlap functions are again expressed in terms of $O(3)$ Clebsch–Gordan coefficients [7]. The corresponding interbase expansion is

$$\Psi_{Jlm}(\theta_1, \theta_2, \theta_3) = \sum_{n=-(J-|m|)}^{J-|m|} (-\mathrm{i})^{l-|m|} C^{l,|m|}_{\frac{1}{2}J,\frac{1}{2}(|m|+n);\frac{1}{2}J,\frac{1}{2}(|m|-n)} \Psi_{Jmn}(\theta_1', \theta_2', \theta_3) \qquad (6.19)$$

where n has the same parity as $(J - |m|)$ and

$$\begin{aligned} u_0 &= R\cos\theta_1\cos\theta_2\cos\theta_3 = R\cos\theta_1'\cos\theta_3 \\ u_1 &= R\cos\theta_1\cos\theta_2\sin\theta_3 = R\cos\theta_1'\sin\theta_3 \\ u_2 &= R\cos\theta_1\sin\theta_2 = R\sin\theta_1'\cos\theta_2' \\ u_3 &= R\sin\theta_1 = R\sin\theta_1'\sin\theta_2' \end{aligned} \qquad (6.20)$$

(see figure 12). The corresponding hyperspherical function is

$$\Psi_{Jlm}(\theta_1, \theta_2, \theta_3) = \frac{\sqrt{(2J+1)(J+l+1)!(J-l)!}}{2^{l+1}\Gamma(J+\frac{3}{2})} (\cos\theta_1)^l P_{J-l}^{(l+\frac{1}{2}, l+\frac{1}{2})}(\sin\theta_1) Y_{lm}\left(\tfrac{1}{2}\pi - \theta_2, \theta_3\right)$$

and the wavefunction $\Psi_{Jmn}(\theta_1', \theta_2', \theta_3)$ is given by equation (6.3) (with n replaced by m).

The contraction in this case (see figure 12 and equation (6.22) below) will involve three quantum numbers J, l and m. Equation (6.7) expressing Clebsch–Gordan coefficients in terms of the ${}_3F_2$ function is not convenient for taking this limit. Instead, we use the following integral representation [7]:

$$\begin{aligned} C^{l,|m|}_{\frac{1}{2}J,\frac{1}{2}(|m|+n);\frac{1}{2}J,\frac{1}{2}(|m|-n)} &= (\mathrm{i})^{l-|m|}(-1)^{\frac{1}{2}(J-|m|-n)} \\ &\times \left\{ \frac{(l+|m|)!(\frac{1}{2}(J-|m|-|n|))!(\frac{1}{2}(J-|m|+|n|))!}{(l-|m|)!(\frac{1}{2}(J+|m|-|n|))!(\frac{1}{2}(J+|m|+|n|))!} \right\}^{1/2} \\ &\times \frac{\sqrt{(2l+1)(J-l)!(J+l+1)!}}{2^{l+|m|+2}\Gamma(J+3/2)} \\ &\times \frac{1}{\sqrt{\pi}} \int_0^{2\pi} (\sin\phi)^{l-|m|} P_{J-l}^{(l+\frac{1}{2}, l+\frac{1}{2})}(\cos\phi)\, \mathrm{e}^{-\mathrm{i}n\phi}\, \mathrm{d}\phi \end{aligned} \qquad (6.21)$$

and the formulae [19]

$$P_n^{(\alpha,\alpha)}(\cos\phi) = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)n!} \times \begin{cases} {}_2F_1(-\frac{1}{2}n, \frac{1}{2}(n+1)+\alpha; \alpha+1; \sin^2\phi) & n \text{ even} \\ \cos\phi\, {}_2F_1(-\frac{1}{2}(n-1), \frac{1}{2}n+\alpha+1; \alpha+1; \sin^2\phi) & n \text{ odd.} \end{cases}$$

After integrating over ϕ, we obtain a representation of the Clebsch–Gordan coefficients in terms of the hypergeometrical function ${}_4F_3$ (of argument 1):

$$C^{l,|m|}_{\frac{1}{2}J,\frac{1}{2}(|m|+n);\frac{1}{2}J,\frac{1}{2}(|m|-n)} = (\mathrm{i})^{l-|m|}(-1)^{\frac{1}{2}(J-|m|)-n} \frac{\sqrt{2l+1}}{2^{2l}}$$

$$\times\sqrt{\frac{(J+l+1)!(\frac{1}{2}(J-|m|-|n|))!(\frac{1}{2}(J+|m|-|n|))!}{(J-l)!(\frac{1}{2}(J-|m|+|n|))!(\frac{1}{2}(J+|m|+|n|))!}}$$

$$\times\begin{cases}\dfrac{\sqrt{(l-|m|)!(l+|m|)!}}{\Gamma(1+\frac{1}{2}(l+|m|-n))\Gamma(1+\frac{1}{2}(l-|m|-n))}\dfrac{\Gamma\left(\frac{1}{2}(J-l+1)\right)}{\Gamma(\frac{1}{2}(J+l+3))} & \\ \quad\times{}_4F_3\left(\begin{matrix}-\frac{1}{2}n,-\frac{1}{2}(n-1),\frac{1}{2}(J+l)+1,-\frac{1}{2}(J-l)\\ \frac{1}{2},1+\frac{1}{2}(l-|m|-n),1+\frac{1}{2}(l+|m|-n)\end{matrix}\middle|1\right) & (J-l)\text{ even}\\ \dfrac{-\mathrm{i}n\sqrt{(l-|m|)!(l+|m|)!}}{\Gamma(\frac{1}{2}(l+|m|-n+3))\Gamma(1+\frac{1}{2}(l-|m|-n+3))}\dfrac{\Gamma\left(\frac{1}{2}(J-l)\right)}{\Gamma(\frac{1}{2}(J+l+2))} & \\ \quad\times{}_4F_3\left(\begin{matrix}-\frac{1}{2}(n-1),-\frac{1}{2}(n-2),\frac{1}{2}(J+l+3),-\frac{1}{2}(J-l-1)\\ \frac{3}{2},\frac{1}{2}(l-|m|-n+3),\frac{1}{2}(l+|m|-n+3)\end{matrix}\middle|1\right) & (J-l)\text{ odd.}\end{cases}$$

(To the best of our knowledge, this expression is new.) In the contraction limit $R\to\infty$ and

$$\begin{matrix}\theta_1\sim\frac{x_3}{R} & \theta_2\sim\frac{x_2}{R} & \theta_3\sim\frac{x_1}{R} & \theta_1'\sim\frac{\rho}{R}\\ J\sim kR & l\sim pR & m\sim k_1R & \end{matrix}\tag{6.22}$$

we obtain

$$\lim_{R\to\infty}(-1)^{-\frac{1}{2}(J-|m|)}\Psi_{Jlm}(\theta_1,\theta_2,\theta_3)=\sqrt{\frac{2kp}{\pi k_2k_3}}\frac{\mathrm{e}^{\mathrm{i}k_1x_1}}{\pi}$$

$$\times\begin{cases}\cos k_2x_2\cos k_3x_3 & (J-|m|)\text{ even, }(l-|m|)\text{ even}\\ -\mathrm{i}\sin k_2x_2\cos k_3x_3 & (J-|m|)\text{ odd, }(l-|m|)\text{ even}\\ -\mathrm{i}\cos k_2x_2\sin k_3x_3 & (J-|m|)\text{ even, }(l-|m|)\text{ odd}\\ -\sin k_2x_2\sin k_3x_3 & (J-|m|)\text{ odd, }(l-|m|)\text{ odd}\end{cases}\tag{6.23}$$

$$\lim_{R\to\infty}(-\mathrm{i})^{l-|m|}(-1)^{-\frac{1}{2}(J-|m|)}\sqrt{R}C^{l,|m|}_{\frac{1}{2}J,\frac{1}{2}(|m|-n);\frac{1}{2}J,\frac{1}{2}(|m|+n)}=\sqrt{\frac{8p}{(k^2-k_1^2)\pi}}(\sin 2\phi)^{-1/2}$$

$$\times\begin{cases}\cos n\phi & (J-|m|)\text{ even, }(l-|m|)\text{ even}\\ -\mathrm{i}\sin n\phi & (J-|m|)\text{ odd, }(l-|m|)\text{ even}\\ -\mathrm{i}\sin n\phi & (J-|m|)\text{ even, }(l-|m|)\text{ odd}\\ -\cos n\phi & (J-|m|)\text{ odd, }(l-|m|)\text{ odd}\end{cases}\tag{6.24}$$

where $\cos\phi=(p^2-k_1^2)/(k^2-k_1^2)$ and $k^2=p^2+k_3^2=k_1^2+k_2^2+k_3^2$. Substituting the formulae (6.5), (6.23) and (6.24) into the expansion (6.19) we obtain

$$\begin{Bmatrix}\cos k_2x_2\cos k_3x_3\\ \sin k_2x_2\cos k_3x_3\\ \cos k_2x_2\sin k_3x_3\\ \sin k_2x_2\sin k_3x_3\end{Bmatrix}=\sum_{n=-\infty}^{\infty}\begin{Bmatrix}\cos n\phi\\ \sin n\phi\\ \sin n\phi\\ \cos n\phi\end{Bmatrix}J_{|n|}(q\rho)\,\mathrm{e}^{\mathrm{i}n\theta_2'}$$

where

$$\tan\theta_2'=\frac{x_3}{x_2}\qquad q^2=k_2^2+k_3^2\qquad \rho^2=x_2^2+x_3^2\qquad \cos^2\phi=\frac{k_2^2}{k_2^2+k_3^2}.$$

These results are special cases of equation (4.31) and (4.32) of section 4.4.

7. Conclusions

The 'method of trees' was introduced [2–7] in order to describe the separation of variables on homogeneous spaces of compact Lie groups, more specifically $O(n+1)$ and $SU(n)$. The 'trees' turned out to be related to subgroup chains and it is useful to complement the tree diagrams by subgroup diagrams [2]. Moreover, the method of trees has been extended in a simple and straightforward way to Euclidean spaces [2], where instead of trees we have clusters of trees. The S_n tree diagrams and E_n cluster diagrams are very helpful in the study of contractions. They tell us, at least for subgroup-type coordinates, how coordinates on S_n and E_n can be related by contractions.

The contribution of this paper is to treat contractions of interbase expansions and hence of the overlap function. Overlap functions for different bases corresponding to isomorphic subgroup chains involve rotation matrices. If the subgroup chains are not isomorphic, the overlap functions will be expressed in terms of Clebsch–Gordan coefficients, Racah coefficients or higher-order recoupling coefficients. For all of these we obtain asymptotic expressions.

For $O(3)$ the contraction breaks the equivalence of the two types of subgroup chains. One $O(3) \supset O(2)$ basis contracts to an $E(2) \supset O(2)$ basis and the other to an $E(2) \supset E(1) \otimes E(1)$ one. Thus we obtain the well known relations between plane and cylindrical waves in E_2.

For $O(4)$ the contraction provides relations between spherical and cylindrical bases and also between cylindrical and Cartesian ones. The interbase expansions relating $E(3)$ Cartesian and spherical bases are obtained by composing the elementary transitions.

Work is in progress on an extension of the methods of this paper and of [1] to spaces of negative constant curvature, generalizing the results on the two-dimensional hyperboloid, obtained earlier [25].

In this paper we restricted ourselves to subgroup-type coordinates only. Two earlier articles were devoted to contractions of separated basis functions that correspond to non-subgroup-type coordinates, in particular, elliptic coordinates on S_2 and on the hyperboloid H_2 [1, 25]. It would also be possible to obtain interbase expansions for other types of bases, though so far this has not been done.

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Appendix

Let us consider all particular cases of the overlap functions T. The substitution $S_{\alpha_j} = -1$ $\alpha_j = 0, 1$ in the general formula (2.7), where $\alpha_i = \alpha, \beta, \gamma$ gives us *two* versions of the T-coefficients. Using several times the formula for the hypergeometric function ${}_4F_3$ [19]

$$
{}_4F_3\left\{\begin{matrix} -n, c, d, b; \\ e, f, g \end{matrix}\middle| 1\right\} = \frac{(f-b)_n (g-b)_n}{(f)_n (g)_n}\, {}_4F_3\left\{\begin{matrix} -n, b, e-c, e-d; \\ e, 1+b-n-f, 1+b-n-g \end{matrix}\middle| 1\right\}
$$

$$
-n+b+c+d = -1+e+f+g \tag{A.1}
$$

one can show that these two variants can be united into *one* formula. We mention here that some formulae for the T-coefficients [5, 7] must be prepared for contractions by using (A.1).

Let us list seven formulae for the T-coefficients adapted to the contractions.

Open end α. (See figure 3(*b*).) Putting $S_\alpha = -1$ and $\alpha = 0, 1$ in formula (2.7) and using the transformation (A.1), we obtain

$$
\begin{aligned}
T^{\beta\gamma}_{Jlm} = {}& \left[\tfrac{1}{2}(1 + (-1)^{J-l+m-\beta})\right] \frac{2^{l-\beta+\frac{1}{2}S_\gamma}}{\Gamma(\beta + \frac{1}{2}S_\beta + 1)} \\
& \times \sqrt{\frac{(2l + S_\beta + S_\gamma + 2)(2m + S_\beta + 1)(J - l)!}{\Gamma(\frac{1}{2}(J + m + \gamma + S_\beta + S_\gamma + 3))\Gamma(\frac{1}{2}(J + m - \gamma + S_\beta + 3))}} \\
& \times \Big\{ \left[(\tfrac{1}{2}(J - m - \gamma))!(m + \beta + S_\beta)!\Gamma(\tfrac{1}{2}(J - m + \gamma + S_\gamma) + 1) \right. \\
& \left. \times \Gamma(\tfrac{1}{2}(l + \beta + \gamma + S_\beta + S_\gamma) + 1)\Gamma(\tfrac{1}{2}(l + \beta - \gamma + S_\beta) + 1) \right] \\
& \times \left[\Gamma(\tfrac{1}{2}(l - \beta + \gamma + S_\gamma) + 1)(\tfrac{1}{2}(l - \beta - \gamma))!(m - \beta)! \right. \\
& \left. \times (J + l + S_\beta + S_\gamma + 2)! \right]^{-1} \Big\}^{1/2} \frac{\Gamma(\frac{1}{2}(J + m + l - \beta + S_\beta + S_\gamma + 3))}{\Gamma(\frac{1}{2}(J - l - m + \beta) + 1)} \\
& \times {}_4F_3 \left(\left. \begin{matrix} -\frac{1}{2}(m - \beta), -\frac{1}{2}(m - \beta - 1), -\frac{1}{2}(l - \beta + \gamma + S_\gamma), -\frac{1}{2}(l - \beta - \gamma); \\ \beta + \frac{1}{2}S_\beta + 1, -\frac{1}{2}(J + l + m - \beta + S_\beta + S_\gamma + 1), \frac{1}{2}(J - l - m + \beta) + 1 \end{matrix} \right| 1 \right).
\end{aligned}
\tag{A.2}
$$

Open end β. (See figure 3(*c*).) Choosing the parameters $S_\beta = -1$ and $\beta = 0, 1$ in formula (2.7), we arrive at the following form of the coefficients $T^{\alpha\gamma}_{Jlm}$:

$$
\begin{aligned}
T^{\alpha\gamma}_{Jlm} = {}& \frac{[1 + (-1)^{l-\gamma+m-\alpha}]}{4\sqrt{\pi}} (-1)^{\frac{1}{2}(m-\alpha)} \sqrt{(2m + S_\alpha + 1)(2l + S_\gamma + 1)} \\
& \times \Big\{ \left[\Gamma(\tfrac{1}{2}(J - m - \gamma) + 1)\Gamma(\tfrac{1}{2}(J + \alpha + l + S_\alpha + S_\gamma + 3)) \right. \\
& \left. \times \Gamma(\tfrac{1}{2}(J + \alpha - l + S_\alpha) + 1)\Gamma(\tfrac{1}{2}(J + m - \gamma + S_\alpha + 3)) \right] \\
& \times \left[\Gamma(\tfrac{1}{2}(J - m + \gamma + S_\gamma) + 1)\Gamma(\tfrac{1}{2}(J - \alpha - l) + 1) \right. \\
& \left. \times \Gamma(\tfrac{1}{2}(J - \alpha + l + S_\gamma + 3))\Gamma(\tfrac{1}{2}(J + m + \gamma + S_\alpha + S_\gamma + 3)) \right]^{-1} \Big\}^{1/2} \\
& \times \begin{cases} A \cdot {}_4F_3 \left(\left. \begin{matrix} -\frac{1}{2}(m - \alpha), \frac{1}{2}(m + \alpha + S_\alpha + 1), \\ \frac{1}{2}(l - \gamma + 1), -\frac{1}{2}(l + \gamma + S_\gamma); \\ \frac{1}{2}, \frac{1}{2}(J + \alpha - \gamma + S_\alpha + 3), \\ -\frac{1}{2}(J - \alpha + \gamma + S_\gamma) \end{matrix} \right| 1 \right) & (m - \alpha) \text{ even} \\
-2\mathrm{i}B \cdot {}_4F_3 \left(\left. \begin{matrix} -\frac{1}{2}(m - \alpha - 1), \frac{1}{2}(m + \alpha + S_\alpha) + 1, \\ \frac{1}{2}(l - \gamma) + 1, -\frac{1}{2}(l + \gamma + S_\gamma - 1); \\ \frac{3}{2}, \frac{1}{2}(J + \alpha - \gamma + S_\alpha) + 2, \\ -\frac{1}{2}(J - \alpha + \gamma + S_\gamma - 1) \end{matrix} \right| 1 \right) & (m - \alpha) \text{ odd} \end{cases}
\end{aligned}
\tag{A.3}
$$

where

$$
A = \frac{\Gamma\left(\frac{1}{2}(J - \alpha + \gamma + S_\gamma) + 1\right)}{\Gamma\left(\frac{1}{2}(J + \alpha - \gamma + S_\alpha + 3)\right)} \Big\{ \left[\Gamma(\tfrac{1}{2}(l + \gamma + S_\gamma + 1))\Gamma(\tfrac{1}{2}(l - \gamma + 1)) \right.
$$

$$\times\Gamma(\tfrac{1}{2}(m+\alpha+S_\alpha+1)\Gamma(\tfrac{1}{2}(m-\alpha+1))\big]\big[\Gamma(\tfrac{1}{2}(l+\gamma+S_\gamma)+1)$$

$$\times\Gamma(\tfrac{1}{2}(l-\gamma)+1)\Gamma(\tfrac{1}{2}(m+\alpha+S_\alpha)+1)\Gamma(\tfrac{1}{2}(m-\alpha)+1)\big]^{-1}\Big\}^{1/2}$$

$$B=\frac{\Gamma\left(\frac{1}{2}(J-\alpha+\gamma+S_\gamma+1)\right)}{\Gamma\left(\frac{1}{2}(J+\alpha-\gamma+S_\alpha)+2\right)}\Big\{\big[\Gamma(\tfrac{1}{2}(l+\gamma+S_\gamma)+1)\Gamma(\tfrac{1}{2}(l-\gamma)+1)$$

$$\times\Gamma(\tfrac{1}{2}(m+\alpha+S_\alpha)+1)\Gamma(\tfrac{1}{2}(m-\alpha)+1)\big]\big[\Gamma(\tfrac{1}{2}(l+\gamma+S_\gamma+1))$$

$$\times\Gamma(\tfrac{1}{2}(l-\gamma+1))\Gamma(\tfrac{1}{2}(m+\alpha+S_\alpha+1))\Gamma(\tfrac{1}{2}(m-\alpha+1))\big]^{-1}\Big\}^{1/2}.$$

Open end γ. (See figure 3(d).) In this case, the coefficient $T^{\alpha\beta}_{Jlm}$ differs from the coefficient $T^{\beta,\gamma}_{Jlm}$ in (A.2) by the substitutions: $m \Longleftrightarrow l$, $\gamma\to\alpha$, $S_\gamma\to S_\alpha$ and by the phase factor $(-1)^{\frac{1}{2}(J-|m|-l+\beta)}$:

$$T^{\alpha\beta}_{Jlm}=\big[\tfrac{1}{2}(1+(-1)^{J-m+l-\beta})\big](-1)^{\frac{1}{2}(J-|m|-l+\beta)}$$

$$\times\frac{2^{m-\beta+\frac{1}{2}S_\alpha+1}\Gamma(\frac{1}{2}(J+l+m-\beta+S_\alpha+S_\beta+3))}{\Gamma(\beta+\frac{1}{2}S_\beta+1)\Gamma(\frac{1}{2}(J-l-m+\beta)+1)}$$

$$\times\Big\{\big[\Gamma(\tfrac{1}{2}(m+\beta+\alpha+S_\alpha+S_\beta)+1)\Gamma(\tfrac{1}{2}(m+\beta-\alpha+S_\beta)+1)(J-m)!$$

$$\times(\tfrac{1}{2}(J-l-\alpha))!(l+\beta+S_\beta)!\Gamma(\tfrac{1}{2}(J-l+\alpha+S_\alpha)+1)\big]$$

$$\times\big[\Gamma(\tfrac{1}{2}(J+l-\alpha+S_\beta+3))\Gamma(\tfrac{1}{2}(m-\beta+\alpha+S_\alpha)+1)$$

$$\times(\tfrac{1}{2}(m-\beta-\alpha))!(l-\beta)!(J+m+S_\alpha+S_\beta+2)!\big]^{-1}\Big\}^{1/2}$$

$$\times\sqrt{\frac{(m+\frac{1}{2}(S_\alpha+S_\beta)+1)(l+\frac{1}{2}(S_\beta+1))}{\Gamma(\frac{1}{2}(J+l+\alpha+S_\alpha+S_\beta+3))}}$$

$$\times{}_4F_3\left(\begin{matrix}-\frac{1}{2}(l-\beta),-\frac{1}{2}(l-\beta-1),-\frac{1}{2}(m-\beta+\alpha+S_\alpha),-\frac{1}{2}(m-\alpha-\beta);\\ \beta+\frac{1}{2}S_\beta+1,-\frac{1}{2}(J+l+m-\beta+S_\beta+S_\alpha+1),\frac{1}{2}(J-l-m+\beta)+1\end{matrix}\middle|1\right).\tag{A.4}$$

Ends α and β are open. (See figure 3(e).) The corresponding T^{γ}_{Jlm} coefficient has the following form [7]:

$$T^{\gamma}_{Jlm}=(\mathrm{i})^{l-\gamma}(-1)^{\frac{1}{2}(J-l-\gamma)}C^{l+\frac{1}{2}S_\gamma,\gamma+\frac{1}{2}S_\gamma}_{\frac{1}{2}J+\frac{1}{4}S_\gamma,\frac{1}{2}(\gamma+m)+\frac{1}{4}S_\gamma;\frac{1}{2}J+\frac{1}{4}S_\gamma,\frac{1}{2}(\gamma-m)+\frac{1}{4}S_\gamma}$$

$$=(-\mathrm{i})^{l-\gamma}\Big\{\big[\Gamma(\tfrac{1}{2}(J+\gamma-m)+\tfrac{1}{2}S_\gamma+1)(\tfrac{1}{2}(J-\gamma+m))!$$

$$\times(l+\gamma+S_\gamma)!(2l+S_\gamma+1)\big]\big[\Gamma(\tfrac{1}{2}(J+\gamma+m)+\tfrac{1}{2}S_\gamma+1)(\tfrac{1}{2}(J-\gamma-m))!$$

$$\times(J-l)!(J+l+S_\gamma+1)!(l-\gamma)!\big]^{-1}\Big\}^{1/2}\frac{\Gamma(J+\frac{1}{2}S_\gamma+1)}{\Gamma(\gamma+\frac{1}{2}S_\gamma+1)}$$

$$\times{}_3F_2\left(\begin{matrix}-\frac{1}{2}(J-m-\gamma),-l-\frac{1}{2}S_\gamma,l+\frac{1}{2}S_\gamma+1;\\ -J-\frac{1}{2}S_\gamma,\gamma+\frac{1}{2}S_\gamma+1\end{matrix}\middle|1\right)\tag{A.5}$$

where $C^{l,\gamma}_{a,\alpha;b,\beta}$ are the Clebsch–Gordan coefficients for the $SU(1,1)$ group, if S_γ is odd, and $SU(2)$ group for even S_γ.

Ends α and γ are open. (See figure 3(*f*).) Putting $S_\gamma = -1$ and $\gamma = 0, 1$ in formula (A.4), we obtain two values for the coefficient T^{β}_{Jlm} depending on the parity of $(m-\beta)$. Using the transformation (A.1) several times, we obtain

$$\begin{aligned} T^{\beta}_{Jlm} &= \left[\tfrac{1}{2}(1+(-1)^{J-l+m-\beta})\right](-1)^{\frac{1}{2}(J-m-l+\beta)}2^{l+m-2\beta}\frac{\sqrt{(2l+S_\beta+1)(2m+S_\beta+1)}}{\Gamma(\frac{1}{2}(J-l-m+\beta)+1)} \\ &\quad\times\frac{\Gamma(\frac{1}{2}(J+l+m-\beta+S_\beta)+1)}{\Gamma(\beta+\frac{1}{2}S_\beta+1)} \\ &\quad\times\left\{\frac{(l+\beta+S_\beta)!(J-l)!(J-m)!(m+\beta+S_\beta)!}{(m-\beta)!(l-\beta)!(J+l+S_\beta+1)!(J+m+S_\beta+1)!}\right\}^{1/2} \\ &\quad\times{}_4F_3\left(\begin{matrix}-\frac{1}{2}(m-\beta),-\frac{1}{2}(m-\beta-1),-\frac{1}{2}(l-\beta),-\frac{1}{2}(l-\beta-1);\\ \beta+\frac{1}{2}S_\beta+1,-\frac{1}{2}(J+l+m-\beta+S_\beta),\frac{1}{2}(J-l-m+\beta)+1\end{matrix}\middle|1\right). \end{aligned} \tag{A.6}$$

Ends β and γ are open. (See figure 3(*g*).) The corresponding T^{α}_{Jlm} coefficient has the form

$$T^{\alpha}_{Jlm} = (-\mathrm{i})^{m-\alpha+l}(-1)^{\frac{1}{2}(|l|-l)}C^{m+\frac{1}{2}S_\alpha,\alpha+\frac{1}{2}S_\alpha}_{\frac{1}{2}J+\frac{1}{4}S_\alpha,\frac{1}{2}(\alpha+l)+\frac{1}{4}S_\alpha;\frac{1}{2}J+\frac{1}{4}S_\alpha,\frac{1}{2}(\alpha-l)+\frac{1}{4}S_\alpha}. \tag{A.7}$$

The expression for the Clebsch–Gordan coefficients in terms of the ${}_3F_2$ function is not convenient for taking the contraction limit. Instead, we use the following integral representation [24]:

$$\begin{aligned} C^{J,M}_{j,m_1;j,m_2} &= (\mathrm{i})^{J-M}(-1)^{j-m_1}\left\{\frac{(J+M)!(j-m_1)!(j-m_2)!}{(J-M)!(j+m_1)!(j+m_2)!}\right\}^{1/2} \\ &\quad\times\frac{\sqrt{(2J+1)(2j-J)!(2j+J+1)!}}{2^{J+M+2}\Gamma(2j+3/2)} \\ &\quad\times\frac{1}{\sqrt{\pi}}\int_0^{2\pi}(\sin\phi)^{J-M}P^{(J+\frac{1}{2},J+\frac{1}{2})}_{2j-J}(\cos\phi)\,\mathrm{e}^{\mathrm{i}(m_2-m_1)\phi}\,\mathrm{d}\phi \end{aligned}$$

and the formulae [19]

$$\begin{aligned} P^{(\alpha,\alpha)}_n(\cos\phi) &= \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)n!} \\ &\quad\times\begin{cases} {}_2F_1(-\frac{1}{2}n,\frac{1}{2}(n+1)+\alpha;\alpha+1;\sin^2\phi) & n \text{ even} \\ \cos\phi\,{}_2F_1(-\frac{1}{2}(n-1),\frac{1}{2}n+\alpha+1;\alpha+1;\sin^2\phi) & n \text{ odd.} \end{cases} \end{aligned}$$

After integrating over ϕ, we obtain a representation of the Clebsch–Gordan coefficients in terms of the hypergeometric function ${}_4F_3$,

$$\begin{aligned} &C^{m+\frac{1}{2}S_\alpha,\alpha+\frac{1}{2}S_\alpha}_{\frac{1}{2}J+\frac{1}{4}S_\alpha,\frac{1}{2}(\alpha+l)+\frac{1}{4}S_\alpha;\frac{1}{2}J,\frac{1}{4}S_\alpha,\frac{1}{2}(\alpha-l)+\frac{1}{4}S_\alpha,} \\ &\quad= (\mathrm{i})^{m-\alpha}(-1)^{\frac{1}{2}(J-\alpha)-|l|}\frac{\sqrt{2m+S_\alpha+1}}{2^{2m+S_\alpha+1}}\sqrt{(m-\alpha)!(m+\alpha+S_\alpha)!} \\ &\quad\times\sqrt{\frac{(J+m+S_\alpha+1)!(\frac{1}{2}(J-\alpha-|l|))!\Gamma(\frac{1}{2}(J+\alpha-|l|)+\frac{1}{2}S_\alpha)}{(J-m)!(\frac{1}{2}(J-\alpha+|l|))!\Gamma(\frac{1}{2}(J+\alpha+|l|)+\frac{1}{2}S_\alpha)}} \end{aligned}$$

$$
\times \begin{cases} \dfrac{\Gamma\left(\frac{1}{2}(J-m+1)\right)[\Gamma(\frac{1}{2}(J+m+S_\alpha+3))]^{-1}}{\Gamma(1+\frac{1}{2}(m+\alpha-|l|+S_\alpha))\Gamma(1+\frac{1}{2}(m-\alpha-|l|))} & \\ \quad \times {}_4F_3\left(\begin{matrix} -\frac{1}{2}|l|, -\frac{1}{2}(|l|-1), \frac{1}{2}(J+m+S_\alpha)+1, -\frac{1}{2}(J-m); \\ \frac{1}{2}, 1+\frac{1}{2}(m-\alpha-|l|), 1+\frac{1}{2}(m+\alpha-|l|+S_\alpha) \end{matrix}\middle| 1\right) & (J-m) \text{ even} \\ \dfrac{-\mathrm{i}|l|\Gamma\left(\frac{1}{2}(J-m)+1\right)[\Gamma(\frac{1}{2}(J+m+S_\alpha+2))]^{-1}}{\Gamma(\frac{1}{2}(m+\alpha+S_\alpha-|l|+3))\Gamma(\frac{1}{2}(m-\alpha-|l|+3))} & \\ \quad \times {}_4F_3\left(\begin{matrix} -\frac{1}{2}(|l|-1), -\frac{1}{2}(|l|-2), \frac{1}{2}(J+m+S_\alpha+3), \\ -\frac{1}{2}(J-m-1); \\ \frac{3}{2}, \frac{1}{2}(m-\alpha-|l|+3), \frac{1}{2}(m+\alpha-|l|+3) \end{matrix}\middle| 1\right) & (J-m) \text{ odd.} \end{cases} \tag{A.8}
$$

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